THE MECHANICAL SOLUTION OF SIMULTANEOUS EQUATIONS.

BY

JOHN B. WILBUR.*

I. SYNOPSIS.

Many problems encountered by engineers and research investigators in diverse fields are made cumbersome by the necessary solution of large numbers of simultaneous equations. The solution of such equations presents no theoretical difficulty, but as the number of unknowns to be obtained simultaneously, and, consequently, the number of equations to be solved, increases, the labor involved in their solution mounts rapidly.

Early in 1934, research was started at the Massachusetts Institute of Technology which has resulted in the construction of a machine for the solution of linear simultaneous equations with real coefficients. An experimental model of this machine was built in 1934, which, though mechanically crude, was so successful in its operation as to encourage the construction of a larger machine. The second machine, now in operation, was designed for the direct solution of nine or fewer simultaneous equations. It lends itself, moreover, to the solution of a larger number of equations, provided certain restrictions in their form are met. The theory upon which the machine is based is such that a larger machine capable of solving a larger number of equations directly may be built.

This research has been undertaken in the belief that a few such machines at educational or research centers, available for general use, will be of help in advancing progress in engineering and research.

II. THEORY UPON WHICH MACHINE IS BASED.

The theory upon which the machine is based may be explained by reference to the following set of simultaneous equations:

$$a_{11}x_1 + a_{12}x_2 + D_1 = 0, \qquad (1)$$

$$a_{21}x_1 + a_{22}x_2 + D_2 = 0. \qquad (2)$$

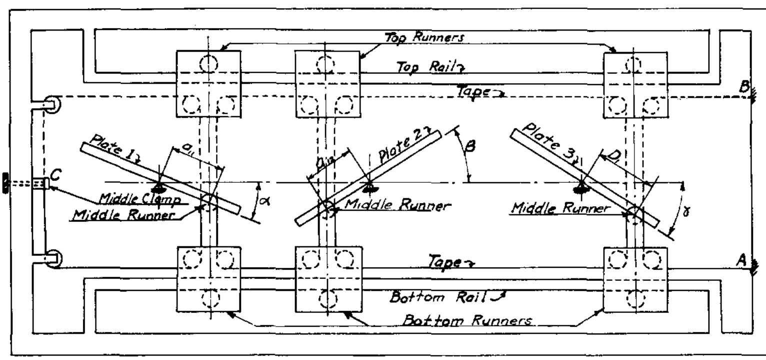
A single equation is reproduced mechanically as shown in Fig. 1. The coefficients a_{11} and a_{12}, and the constant term D_1 are set as distances on tipping plates, to the right or left of the plate pivot depending upon the algebraic sign of these

* Assistant Professor of Civil Engineering, Massachusetts Institute of Technology.

quantities. The equation is represented by a flexible steel tape, fixed at points A and B, and with a fixing clamp at C. That portion of the tape above the tipping plates (shown dotted) has as its function the maintenance of tension in the system during motion, and need not be considered further in an examination of the principles upon which the machine functions. The top and bottom runners, riding on horizontal rails as shown, ensure that the tapes leading away from the tipping plates will always be essentially vertical.

With the quantities a_{11}, a_{12} and D_1 set on their respective plates, let the three plates be leveled, and the clamp at C tightened so that no movement of the tape can occur at that

FIG. 1.



Elevation showing arrangement of plates and tapes for a two-equation machine.

point. If an angular motion is now introduced at any one of the three plates, there are, if only the action of the one tape shown in Fig. 1 is considered, an infinite number of positions which the other plates may assume. For any given position of the plates, however, the following equation holds, assuming that the total length of the tape between points A and C remains unchanged:

$$2a_{11} \sin \alpha + 2a_{12} \sin \beta + 2D_1 \sin \gamma = 0. \qquad (1a)$$

Dividing this equation by $2 \sin \gamma$ we obtain

$$a_{11} \frac{\sin \alpha}{\sin \gamma} + a_{12} \frac{\sin \beta}{\sin \gamma} + D_1 = 0, \qquad (1b)$$

which is equivalent to equation (1) if

$$\frac{\sin \alpha}{\sin \gamma} = x_1 \quad \text{and} \quad \frac{\sin \beta}{\sin \gamma} = x_2.$$

If now two tapes act simultaneously on these same three plates, and in the same manner as that just described, the distances a_{21}, a_{22} and D_2 being set on plates 1, 2 and 3, respectively, for the second tape, and if the same procedure is followed in first leveling all plates, and then introducing a certain angular movement at any one plate, the remaining two plates must undergo a definite corresponding rotation. The sine of the angles through which plates 1 and 2 have rotated give the values of x_1 and x_2 to a certain scale. The actual values of these unknowns are obtained by dividing their relative values by the sine of the angle through which plate 3, upon which the constant terms of the equations have been set, rotates.

III. DESCRIPTION OF MACHINE AS BUILT.

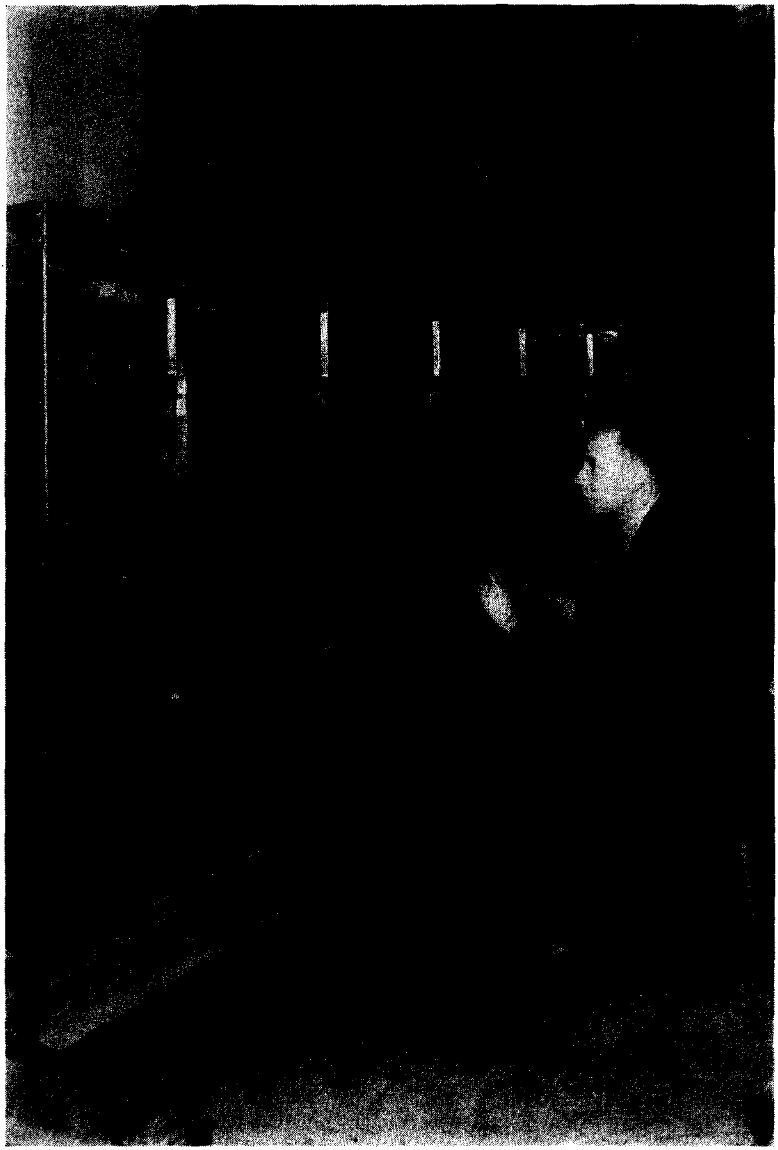
The machine as actually built consists of a heavy steel rectangular frame, within which rotate ten slotted steel plates (see Figs. 2, 3 and 4). Nine of the ten plates represent the nine unknowns for which the machine is built; on the tenth plate the constant terms of the equations are set.

Each plate has ten longitudinal parallel slots. In nine of these slots "middle runners" may be moved in order that the coefficients to the unknowns and the constants of the equations may be set. The tenth slot is utilized by a sine reading mechanism from which the sine of the angle through which a plate rotates may be read directly.

Nine tapes, each about sixty feet in length, represent the nine equations for which the machine is designed. The slots in which are placed the middle runners used for setting coefficients and constants of a given equation, and the tape which represents that equation, all lie in a vertical plane.

The middle runners may be unclamped, and moved freely to their approximate positions for the equations to be solved. Micrometer screws then enable one to make a setting to an accuracy of one two thousandth of an inch, which corresponds to four significant figures on the coefficient and constant scales.

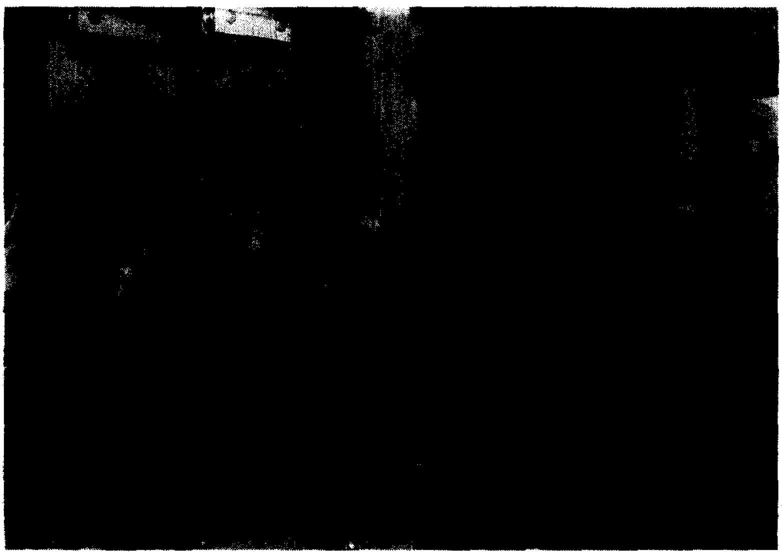
FIG. 2.



Algebra in terms of metal. General view of new machine for solving simultaneous equations.

Inasmuch as the machine reproduces the equations geometrically, careful design and construction were essential. The frame itself is rigid so that it will not substantially distort elastically when operated. In order to minimize tape strains, over one thousand ball bearing pulleys are used, reducing friction to a practical minimum.

FIG. 3.

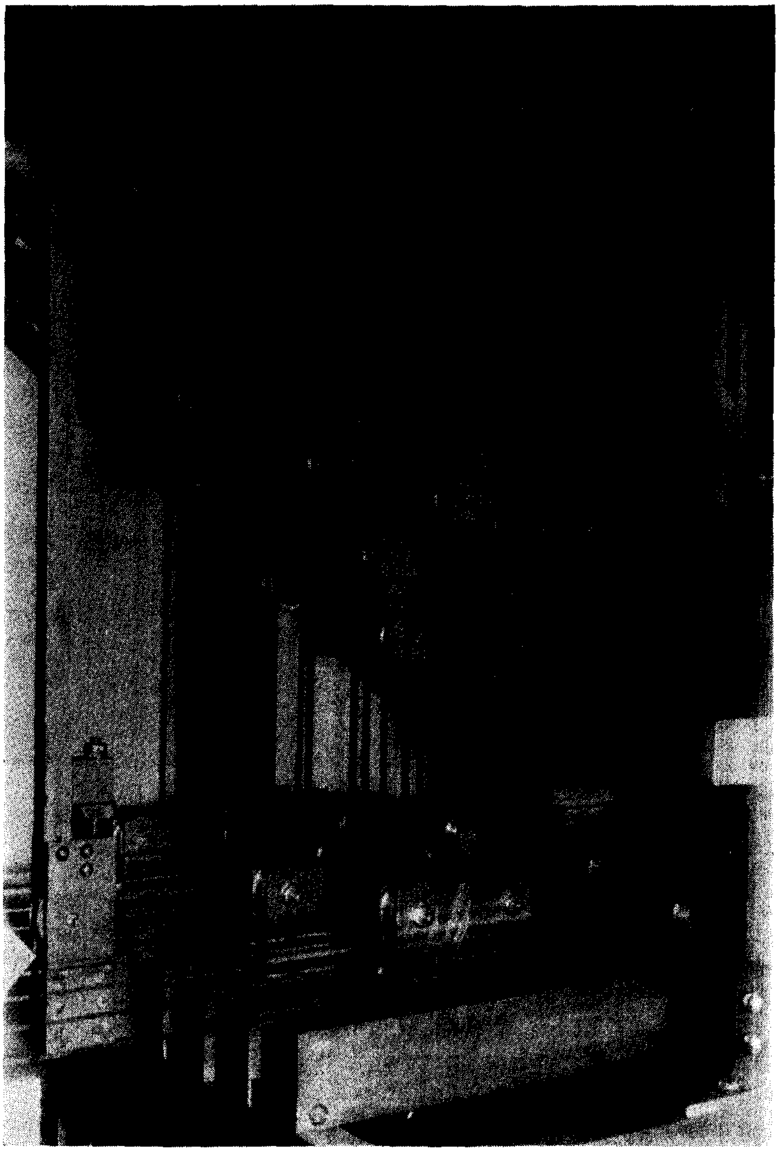


View showing upper side of tipping plate. The middle runners used for setting coefficients or constants on this plate, and corresponding top and bottom runners riding on their rails, may be seen.

IV. OPERATION OF MACHINE.

While the best technique for the operation of the machine has not been definitely established, the following outline will permit an estimate of the labor involved. The middle clamps (at "*C*," Fig. 1) are first loosened, which permits complete freedom of motion within the machine. The plates to be used are then approximately leveled, and all plates are clamped so that they cannot rotate. The coefficients and constants are systematically set on the proper middle runners and their settings checked. In the setting of these middle runners, most of the time required for solution is involved. The setting of a coefficient requires about the same time which

FIG. 4.



View showing under side of tipping plate. The under side of the middle runners for this plate, and the corresponding top and bottom runners riding on their rails, may be seen.

would be needed to turn off an angle on a transit. When these settings are complete, the middle clamps are tightened, only for the tapes in use, and the plates which are in use are unclamped. In this way it is possible to utilize only the portion of the machine required. Any of the plates involved in the solution of the given equations may now be rotated freely, but those plates which will undergo the largest rotations during solution will turn the most easily. Such a plate is found by trial, and used as the driving plate. This plate is rotated clockwise about 40 degrees, and brought to rest with an oscillating motion which tends to eliminate the detrimental effect of friction in the machine. As the driving plate is rotated, the other participating plates also rotate. The sine mechanism is then read on each participating plate. Driving the machine from the same plate, the plates are then brought back through their horizontal position until approximately the same motion has taken place in the opposite direction from that originally introduced, and again brought to rest with an oscillating motion. The sine mechanisms are again read. The value of any given unknown is then given by dividing the algebraic difference of the readings of the sine mechanism for the plate representing that unknown, by the algebraic difference of the readings from the sine mechanism for the plate upon which the constants have been set. Once the coefficients and constants have been set on the machine, the actual operation and interpretation of results require only a few minutes.

The total time required to solve nine simultaneous equations on the machine, to an accuracy of three significant figures, is estimated at from one to three hours, depending upon the equations themselves. This time will undoubtedly be reduced when more experience has been acquired in the technique of operation.

The solution of similar equations employing a keyboard calculator might be expected to require in the neighborhood of eight hours.

V. RESULTS OBTAINED FROM MACHINE.

The accuracy obtained in a given solution will depend upon the equations involved. Certain sets of equations, bordering upon instability, will be difficult to solve, regardless of the method used, while other sets will yield excellent accuracy with little effort. The following two conclusions have however been reached:

1. For the usual set of equations, the errors in the values of the unknowns as given by the machine will not exceed one per cent. of the largest unknown, as read directly from the machine after one operation. In many cases, much better accuracy will result.

2. By successive approximations, arbitrary accuracy can be obtained. The machine is well suited to this procedure. If the results obtained from the first solution are substituted in the original equations and the corresponding constants computed, they will in general differ by a small amount from the actual constants of the equation. This difference is easily obtained with a computing machine. If now the equations are solved again, using in place of the constant terms in the original equations the differences in the constants thus obtained, the new solution yields the corrections to be applied to the unknowns as determined from the first solution. This procedure involves the re-setting of only the constant terms on the machine, hence progresses rapidly. It may be repeated successively to give results to any degree of accuracy required.

An important factor which greatly increases the scope of the machine is the development of a method whereby more than nine equations can be solved by the machine. This expansion is limited by the form of equations which can be thus solved, but it is nevertheless applicable to many important problems. The usual type of simultaneous equations encountered by the structural analyst in the investigation of stresses in an indeterminate structure furnishes an example of the type which can be solved in this manner by the machine. In the development of this use of the machine, eighteen simultaneous equations resulting from a slope deflection solution of a building frame acted upon by lateral loads were solved with an accuracy of between four and five significant figures. It is estimated that under the proper conditions a machine solution of this kind would take seven and one-half hours. The time required for a corresponding solution by a keyboard calculator is estimated at 32 hours. As the number of unknowns increases, the time saved by mechanical solution becomes increasingly important.

VI. FUTURE INVESTIGATIONS.

Our research on this type of mechanism is by no means complete. Minor changes in the present machine are contemplated. More exhaustive tests will undoubtedly show that the tapes now on the machine have neither the optimum flexibility nor initial tension. The technique of operation is as yet imperfect. There is every reason to believe that the accuracy as obtained from a single set of readings can be improved.

The relative success of the present machine indicates that a machine capable of directly solving a larger number of equations is feasible. The development of an automatic means of setting coefficients and constants would add greatly to the practicability of a larger machine, leading to large savings in time of solution.

VII. ACKNOWLEDGMENT.

In a development of this kind, the one responsible for the continuity of effort, while contributing to the final results, acts largely as a clearing house for the ideas of those who work with him. Space does not permit complete acknowledgment of the help that has been given. To Dr. Vannevar Bush, Vice-President and Dean of Engineering at M. I. T., however, much of the credit for the success of this enterprise is due. His original interest in the research, his contribution of several basic ideas, together with his continued interest and suggestions, have made the development possible. To Sir Douglas Alexander, President of the Singer Sewing Machine Company, grateful acknowledgment is tendered for having made the construction of the new machine possible. To Professor Charles B. Breed, Head of the Civil Engineering Department at M. I. T. under which this research was conducted, the writer owes a debt for his interest and friendly encouragement.

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