Chapter 5

Analog Computing Devices

Introduction

Imagine that you are standing on the bank of a small river. On the opposite bank, on a small rise inaccessible to you, is a tall building whose height you would like to determine. Fortunately, you have with you a protractor, or similar angle measuring device, that enables you to sight the angle above the horizontal of the foot and top of the building. Then, turning, you carefully pace a convenient distance away from the building (the ground also being conveniently flat) and repeat the angle measurements. You now have sufficient information to determine the height of the building.

But how do you do the actual calculation? One method is to use trigonometry—develop the formulas that apply in this situation and use a pocket calculator to evaluate them substituting your observed angles and distance paced for the unknowns. An alternative approach would be to do a careful scale drawing (Figure 5.1) from which the height of the building could simply be measured without any knowledge of trigonometry.

The first method of calculation uses a *digital* technique. The quantities involved are represented by numbers (strings of decimal digits—hence the name), and the numbers are manipulated in an abstract manner independent of the original problem.

The second method of calculation uses an *analog* technique. The

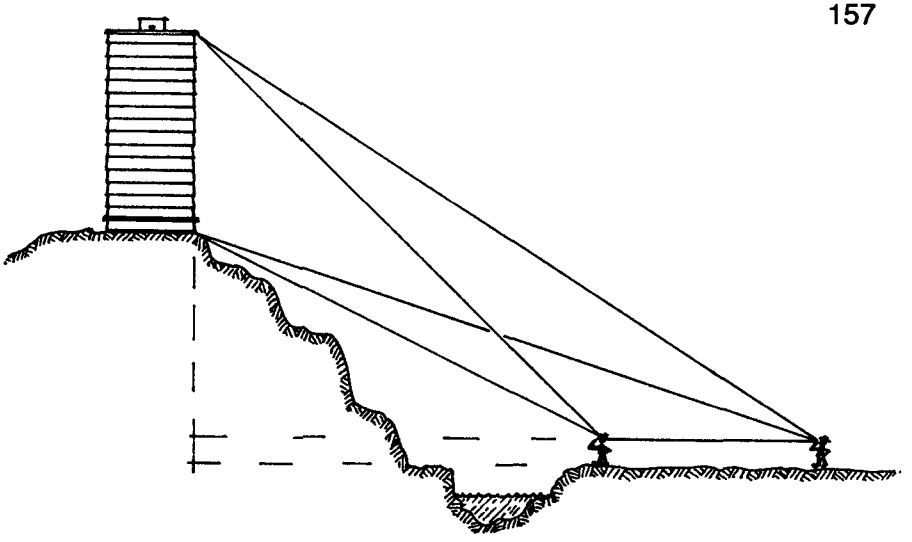


Figure 5.1. A graphical solution to the surveying problem described in the text.

quantities in the problem are represented by a direct proportion (or analog) of the length of lines and the angles between them. Unlike the digital technique, the accuracy of the analog technique is limited by how carefully and accurately the drawing is made and the result measured. On the other hand, the analog technique is generally quicker to apply and less prone to error, as the whole problem is set before you as a picture. Its adaptability is clear if you were to subsequently ask how far away is the building, or how high is the rise on which it stands?

Analog methods of calculation have a very old tradition and many are still in common use. For example, in artillery surveying, digital (computer) techniques are used for the basic calculation and a graphical analog device is still commonly used as a check against error. In World War II, graphical devices were the basic technique. Figure 5.2 shows a plotter used in antiaircraft defense. From an observation post the direction (bearing), angle above the horizon (altitude), and distance to the aircraft (range) can be measured. These are set up on the plotter instrument, and the aircraft's height can then be read off and its position over the ground marked on the map.

Simple direct analogs, such as those just discussed, are very common. There are more sophisticated approaches in which it is not the problem itself that is modeled. Rather, the equations describing the problem are derived and are then modeled in such a way that the original problem is much less evident. It is this approach that is discussed in this chapter.

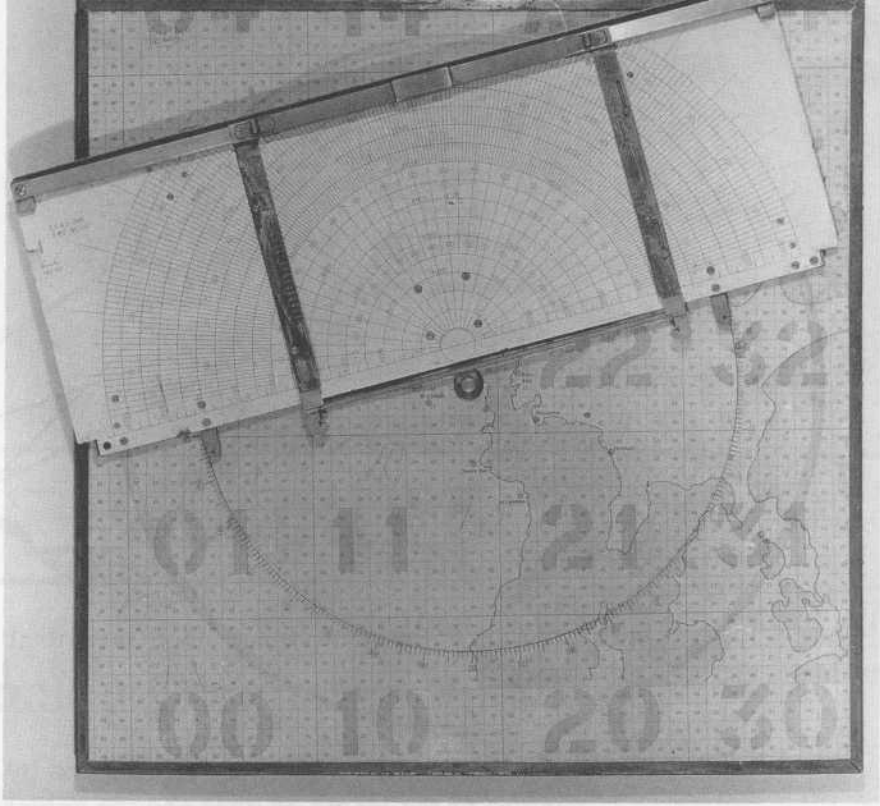


Figure 5.2. An antiaircraft plotter. The vertical triangle, from which the height and horizontal range can be determined from the slant range and altitude angle, is solved by the gridded chart that has been laid flat onto the map.

Simple Analog Devices

We start with some simple examples of analog devices. Figure 5.3 shows a more refined example of a plotting instrument for antiaircraft defense. The altitude angle and range are entered by positioning the angular arm that represents the line of sight to the aircraft. By manually setting the vertical arm the height and horizontal range can be read off.

Figure 5.4 shows a more elaborate mechanism, called a *resolver*, for converting from polar to rectangular coordinates. This mechanism works automatically and is a component part of many of the devices described later. They are found in naval gunnery computers from World War I, and a simpler form occurs in Kelvin's harmonic analyzer in the 1870s and in numerous other harmonic analyzers from the turn of the century onwards.

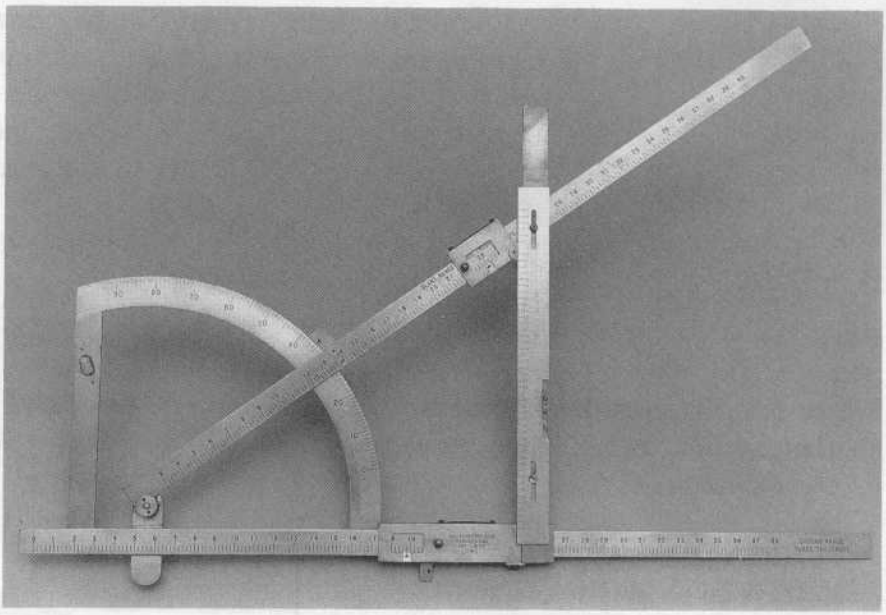


Figure 5.3. A more sophisticated mechanism for solving the vertical triangle in antiaircraft gunnery. Here each of the elements of the triangle is represented by a metal bar that can be rotated or slid to place it in correct relative physical relationship to the elements of the original problem.

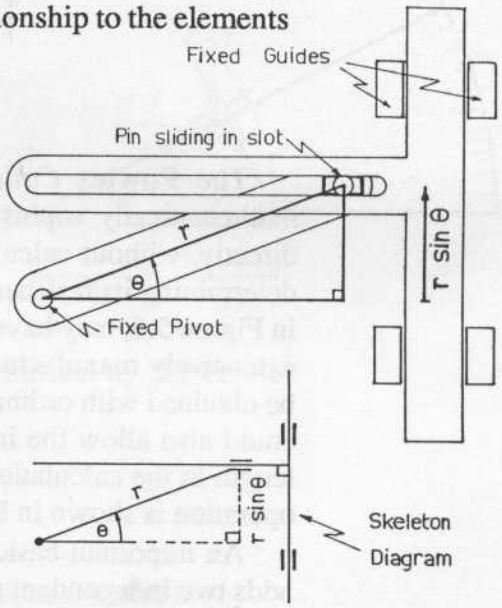


Figure 5.4. A mechanical resolver for converting from polar to rectangular coordinates. An arm is rotated through an angle θ. Sliding radially on the arm is a block carrying a pin whose distance from the center is r. Together r and θ represent the hypotenuse of a right-angle triangle. The pin moves in a horizontal slot in an arm constrained by guides to slide vertically. The vertical movement of the arm is therefore $r \sin \theta$. A similar arm, with a vertical slot, sliding horizontally will yield $r \cos \theta$.

A multiplying mechanism can be constructed using similar triangles, as shown in Figure 5.5. One slide input rotates an arm about a fixed pivot for the first operand. A second slide input positions a slider on the rotated arm for the second operand. The vertical position of the slider on the rotating arm yields the product on an output slide. This multiplier mechanism also appears in World War I gunnery computers, but antecedents are found in some of the more elaborate integraphs of the late nineteenth century.

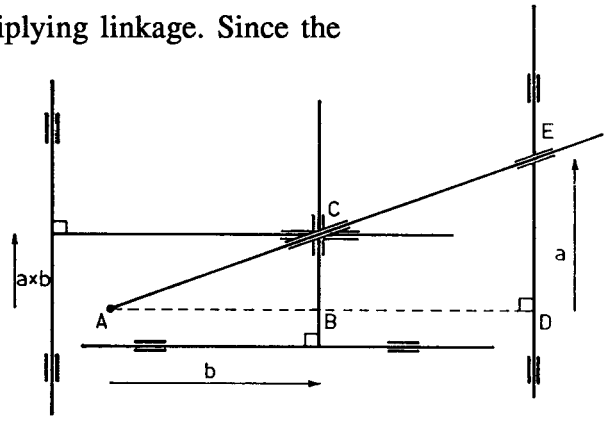


Figure 5.5. Skeleton diagram of a multiplying linkage. Since the triangles *ABC* and *ADE* are similar,

$$BC/AB = DE/AD,$$

so

$$BC = [DE \times AB]/AD = a \times b$$

if *AD* is taken as unit length.

The Powles *Calliparea*, patented about 1870, is a more mathematically sophisticated device. It was designed to measure directly, without calculation, the cross-sectional area of a wire for determining its resistance to electric current. The instrument, shown in Figure 5.6, may have been a prototype model. It was probably not extensively manufactured because the same accuracy in area could be obtained with ordinary callipers and a slide rule, the use of which could also allow the inclusion of the resistivity of the wire and its length in the calculation. The mathematical principle underlying its operation is shown in Figure 5.7.

An important basic analog mechanism is the differential, which adds two independent motions. Several forms of this mechanism are shown in Figure 5.8, which indicates how the common bevel gear differential might be understood from simpler forms. Here it is not the linear position of a slide, but the angle though which a shaft rotates, that provides the analog of a quantity in the original problem.

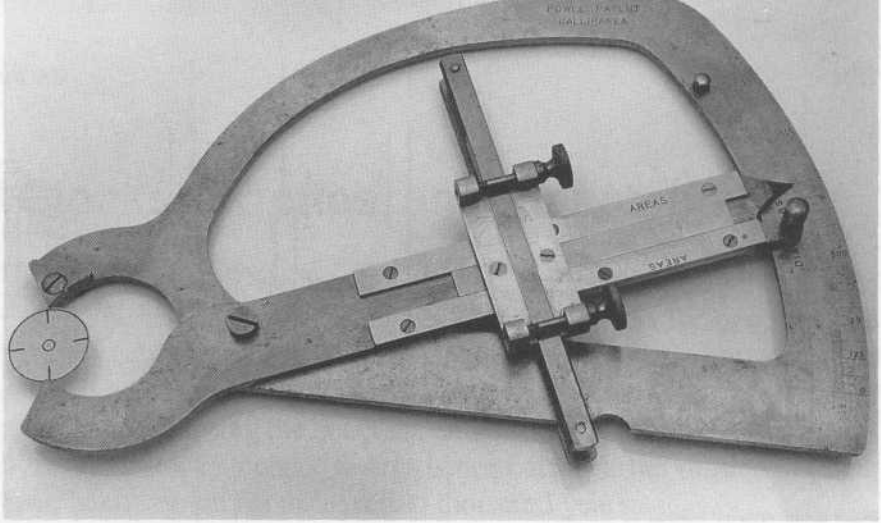


Figure 5.6. Powles Patent Calliparea (ca. 1870) for directly measuring the cross-sectional area of round wire.

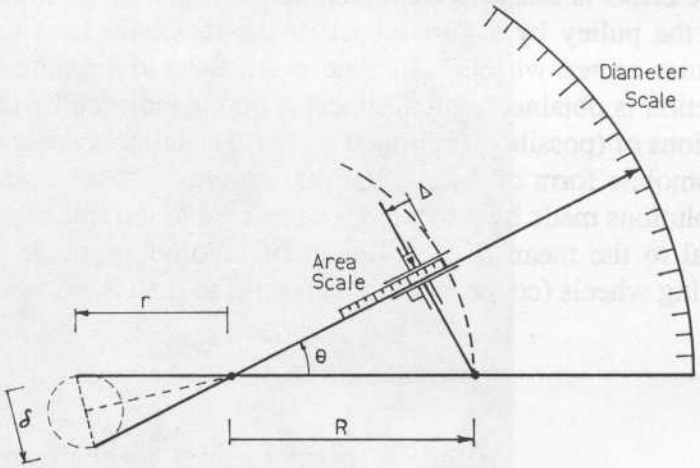


Figure 5.7. The mathematical functions performed by the Powles Calliparea. From the figure,

$$\delta = 2r \sin (\theta/2),$$

$$\Delta = R(1 - \cos \theta) = 2R \sin^2 (\theta/2)$$

by the trigonometric identity,

$$\cos \theta = 1 - 2 \sin^2 (\theta/2)$$

in the instrument $r = 2$ inches and $R = \pi$ inches, where

$$\Delta = 2R(\delta/2r)^2 \quad = \pi/8 \, \delta^2 \quad = 1/2(\pi\delta^2/4)$$

Thus the difference δ represents the cross-sectional area of a circle with diameter δ to a linear scale of 0.5 inch to each square inch of area.

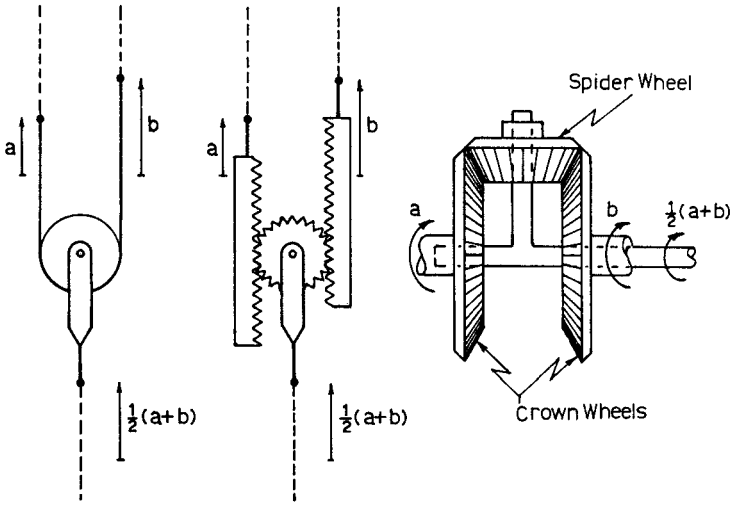


Figure 5.8. Three forms of differential mechanism for adding two independent quantities. If either suspension rope of the pulley at the left is raised, the pulley is raised by half that amount, and the two ropes may be manipulated independently. In the center the same effect is obtained by replacing the ropes by toothed racks and the pulley by a gear wheel. If the racks are bent to form circular crown wheels, as at the right, the same mathematical function is obtained with all linear motions replaced by circular motions of (possibly) unlimited extent. The latter is the common automobile form of differential that ensures that the number of revolutions made by the engine (connected to the spider shaft) is equal to the mean of the number of revolutions made by the driving wheels (connected via half-axles to the crown wheels).

Although a differential mechanism occurred in antiquity (the Antikythera Mechanism, described below), it was not reinvented until the mid-seventeenth century. Its first use in calculating machinery was in the naval gunnery computers of World War I, but a simpler form was used in Kelvin's tide predictors (1870s) and harmonic synthesizers.

All of the devices thus far described are theoretically exact in their action. Aside from "noise" introduced by the limited accuracy in machining the parts, the mechanisms do not introduce any structural error due to the geometry of the mechanism being inexact. Frequently, however, the mathematical description of a problem is simplified so that it can be solved by a more straightforward analog device if the error thus introduced is of no importance to the user.

Astronomical Clocks, Orreries, and Planetariums

The motions of the heavenly bodies, and particularly the planets or "wandering stars," have held a fascination for people since antiquity; this fascination is enshrined in the scientific knowledge of astronomy and the speculative knowledge of astrology. Mechanical models representing the heavens, either to show the present positions of bodies or to "calculate" their positions at an earlier or future epoch, have been known for at least two thousand years.

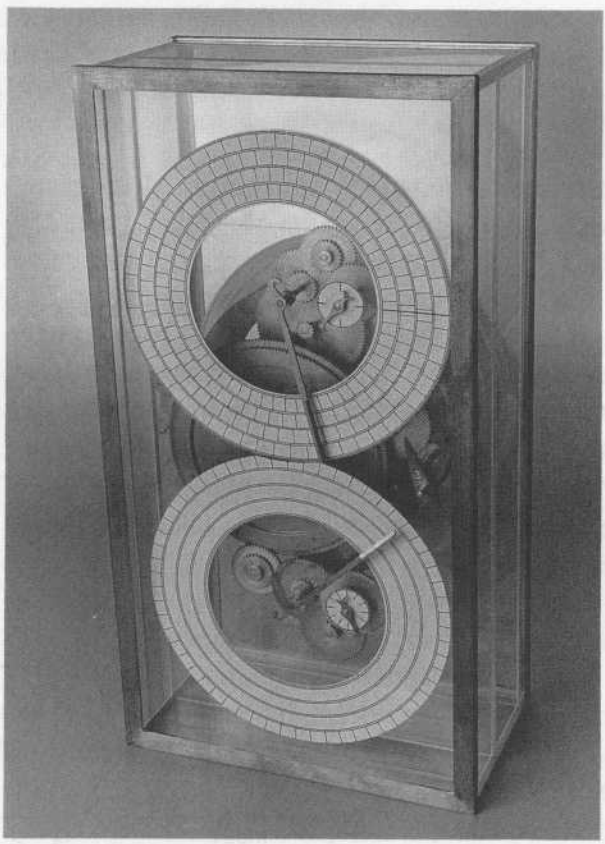
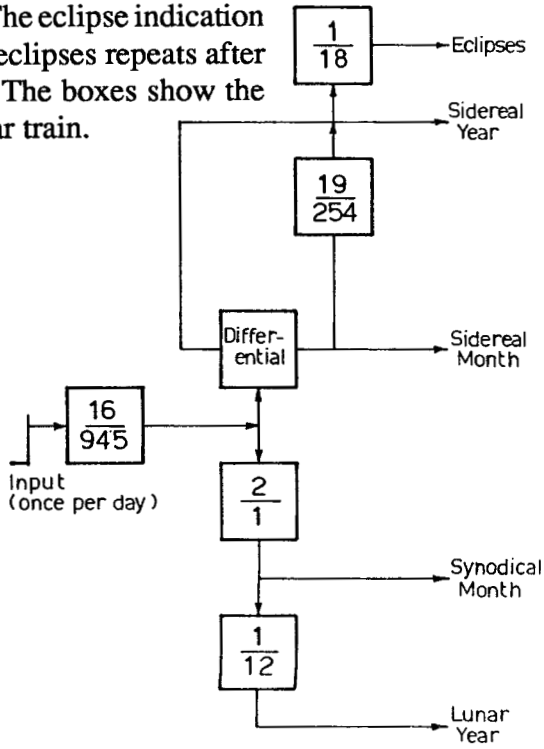


Figure 5.9. The first working modern reconstruction of the Antikythera Mechanism, made by the author. The case is approximately 12"× 6"× 3" (320 × 160 × 80mm). In the original, the case was of bronze and wood, with opening doors like a triptych. The whole was covered by inscriptions in Greek explaining the use of the mechanism.

The earliest extant example of an astronomical computer is the Antikythera Mechanism, from 80-50 B.C., recovered from an ancient wreck in 1901 and interpreted from X-ray images by the historian Derek de Solla Price. Figure 5.9 is a photograph of a modern reconstruction made to a recent reinterpretation of its mechanism and function by the author. The input is turned once per day and drives a dial showing the age of the moon—the 29.53 day cycle from new moon through full moon and back to new moon. This in turn drives dials that show the positions of the sun and the moon among the stars and the eighteen-year cycle of lunar and solar eclipses. Figure 5.10 shows schematically the arrangement of the mechanism which includes thirty-nine gears and is the only example of a differential gear mechanism known before the mid-seventeenth century.

Figure 5.10. The logical organization of the Antikythera Mechanism. The heart of the mechanism is based on the Metonic Cycle: 254 Sidereal Months of 27.32 days (for the moon to return to the same place relative to the fixed stars) takes approximately 235 Synodical Months of 29.53 days (from new moon back to new moon) or 19 Sidereal Years of 365.26 days. The eclipse indication is based on the Saros cycle—the pattern of eclipses repeats after 233 Synodical Months, or nearly 18 years. The boxes show the gear ratios of the various sections of the gear train.



Planispheric astrolabes, which show star and planetary positions in altitude and azimuth coordinates, as seen from the earth's surface, appear to have originated in Hellenic times and Islamic examples are known from the ninth century. Astrolabes were practical tools of navigation and also served as aids to astrological divination and prediction.

With the development of mechanical clocks in the thirteenth century, wheelwork was adapted for directly indicating the motions of heavenly bodies. From the design of Giovanni de Dondi in 1364 the design of astronomical clocks was progressively refined and elaborated. The cathedral clock at Strasbourg (1842) is probably the best known and that of Jens Olsen (Copenhagen, 1955) is arguably the finest. The development of smaller clocks of this type was strongly supported in the eighteenth-century French courts.

Orreries, models showing the heliocentric motions of the planets, are named for one produced for the Earl of Orrey in 1712. They became very well known in the nineteenth century as aids in popular lectures on astronomy. The main element in the design of these instruments was the calculation of geared wheelwork to approximate the astronomical periods involved.

The development of the planetarium in 1919-1923, by Walter Bauersfeld of the Carl Zeiss optical works in Jena, is the most important modern contribution to analog astronomical devices. In this instrument, the firmament and planets are represented by optical projections on the inside of a large hemispherical dome. This instrument has seen considerable elaboration through the twentieth century and features the geocentric representation of planetary motions.

The principle embodied in planetary projection is shown in Figure 5.11. The earth and another planet, say Mars, are represented by the motions of points on circles of a size and inclination to represent the planetary orbits to scale. A rod or similar mechanism joining these two points carries an optical projector for the planetary image on the dome. The planetary orbits, though nearly circular, are in reality ellipses, so the motions are not correctly represented by uniform circular motions. The deviation of the fixed radius of a circle from the ellipse is not of great importance, but the nonuniform motion of the planet in the ellipse, because it accumulates from day to day, cannot be neglected. The linkwork shown in Figure 5.11 is used to approximate the nonuniform planetary motion. Note that this mechanism is not theoretically exact and possesses structural errors.

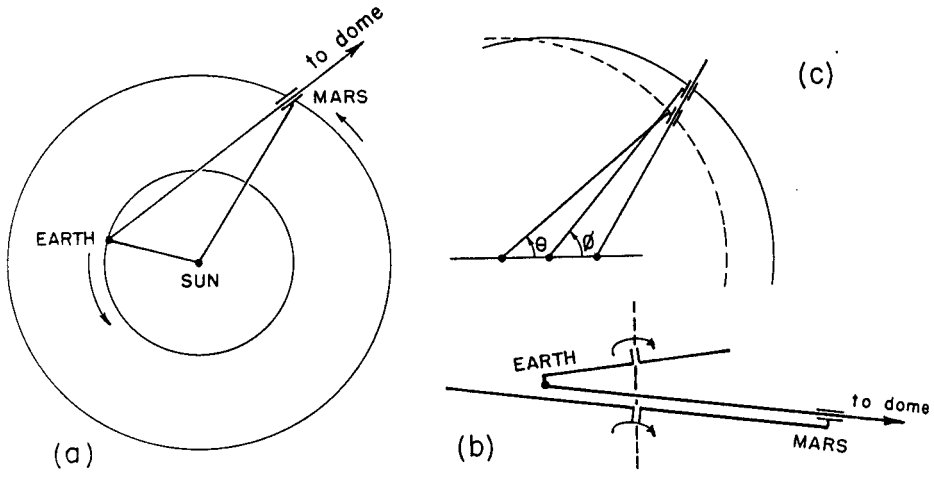


Figure 5.11. The principle employed in locating projected planetary images on the dome of the Zeiss Planetarium. The mechanism models the relative positions of the earth and a planet in space. At (c) is shown the simple link mechanism used to approximate the nonuniform motion of a planet in its elliptical path, ø, from a uniform circular motion, θ.

Its justification lies solely in the adequacy of its precision to the task at hand.

The art of approximating complex functions by simple mechanisms was elaborately developed during World War II by Antonin Svoboda (who later played a leading role in postwar developments of computers in Czechoslovakia). The purpose was to find very simple mechanisms that would produce solutions to complex problems in ballistics and gunnery sufficient for use in military equipment. Commonly, these did not attempt to simplify the mathematical description of the problems but attempted to fit approximate formulas to their solutions.

Planimeters

All the analog devices described thus far have been essentially geometric in nature. The output is dependent only on the present state of the inputs. Previous values of the inputs do not effect the output, and the devices therefore exhibit no memory of past events. We are well aware, however, that much of the power of digital

computers stems from their memory. The same is true of analog devices, much of whose power stems from the discovery in the first half of the nineteenth century of mechanical embodiments of the mathematical function of integration—effectively involving a form of memory of past inputs.[1]

Integrating devices arose first as planimeters—instruments for measuring directly, by tracing the perimeter, the area enclosed by an irregular closed curve such as the boundary of a parcel of land on a map. The idea seems to have been first discovered by the Bavarian engineer Hermann in 1814 and was rediscovered by Tito Gonnella in Florence in 1824, but neither succeeded in having a satisfactory working model made. A further rediscovery by the Swiss, Oppikofer, led to successful manufacture by Ernst in Paris about 1836. In these instruments the integrating wheel moves on the surface of a cone, a principle rediscovered by Sang in 1851 and widely reported in the English literature.

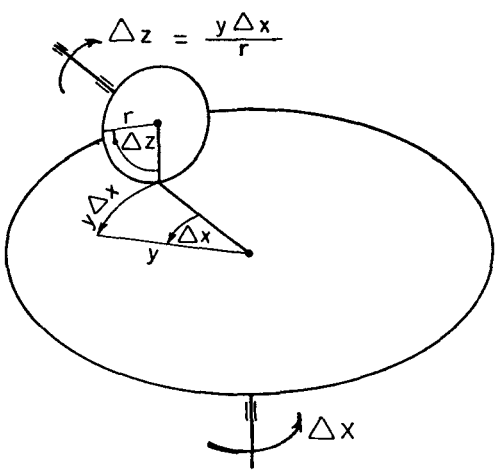
Wetli in Zurich, in 1849, substituted a disc for the cone of the earlier planimeters, making the integration of negative-valued functions possible. The disc-and-wheel integrating mechanism used by Wetli, shown in Figures 5.12 and 5.13, was extensively used in Differential Analyzers in the 1930s and 1940s. In fact, any continuously adjustable, variable-speed drive mechanism can act as an integrator, so a wide variety of different forms exist. Wetli planimeters were manufactured by Starke in Vienna and later improved by Hansen in Gotha.

Figure 5.12. The principle of operation of the Wetli disk-and-wheel integrating mechanism. The dependent variable, y, determines the distance of the integrating wheel from the center of the disk. If the disk is rotated through an angle, Δx, then the integrating wheel is turned by

$$\Delta z = [y\Delta x]/r.$$

Hence, after a period of action,

$$z = 1/r \int y\, dx.$$



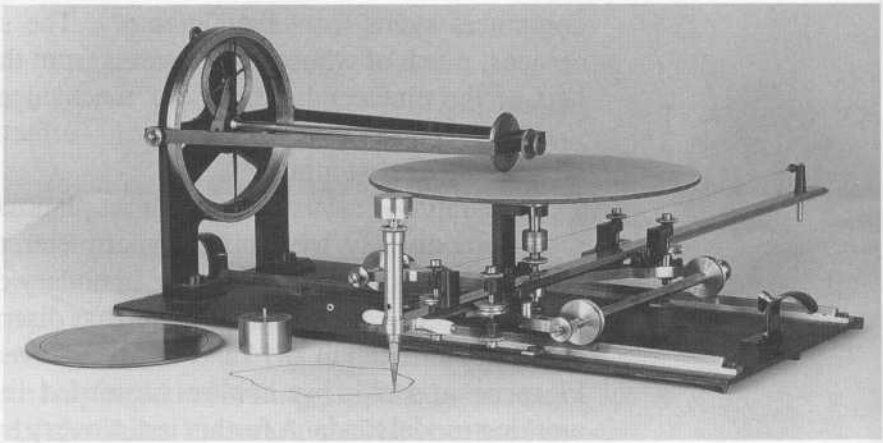


Figure 5.13. A Wetli planimeter (1849). In this planimeter a movement of the tracing point in the x-direction causes the disk to be rotated by means of a flexible wire. Movement in the y-direction causes the disk to be moved on a carriage with the tracing arm so that the integrating wheel is, in effect, displaced from the center of the disk. Courtesy Science Museum, London. Negative No. 3293.

A major step forward was the invention of the polar planimeter in 1856 by Jacob Amsler, then a student at Königsberg. In the Amsler planimeter the integrating wheel moves, by a combination of rolling and sliding motions, over the paper on which the area to be measured is drawn. The principle is shown in Figures 5.14 and 5.15. The simplicity, ease of use, and low price of the Amsler planimeter soon drove all older forms from the field and led to the manufacture of many thousands of the instruments in the nineteenth century—over twelve thousand by Amsler alone by 1884. Amsler-type planimeters are still being manufactured for their traditional uses in surveying, architecture, and engineering design. An important common usage was in the analysis of indicator diagrams to determine the efficiency of steam engines.

Although theoretically exact in its function, the Amsler planimeter is susceptible to faults that limit its precision in practice. From about 1880, several forms of "precision planimeters" were manufactured, particularly by Coradi in Zurich, to reduce these

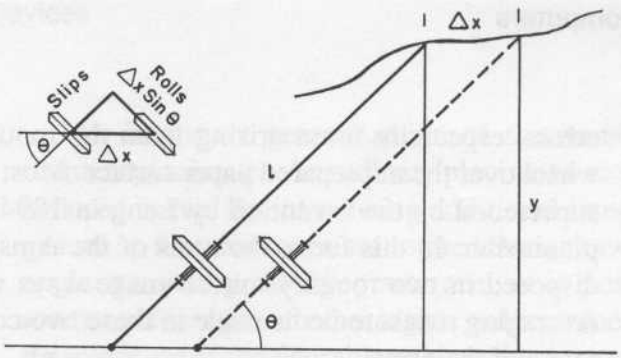


Figure 5.14. The principle of Amsler-type Planimeters. When the ordinate is y, the arm of length l is inclined at an angle $\theta = \sin^{-1}(y/l)$ to the x axis. If the arm translates a distance Δx in the direction of the x axis parallel to itself, the integrating wheel will turn about its axis and slide parallel to its axis. The turning will be through an angle

$$\Delta z = [\Delta x \sin \theta]/r = [y \Delta x]/[lr].$$

The total rotation of the integrating wheel in tracing around a closed curve will therefore be just

$$z = 1/[lr] \int y \, dx.$$

The integrating wheel will also be turned as the arm rotates about its pivot, but in tracing completely around a closed curve the arm will return to its initial position so the net effect of this will be zero. The path that the pivot follows is unimportant and in the Amsler planimeter is just a circle.

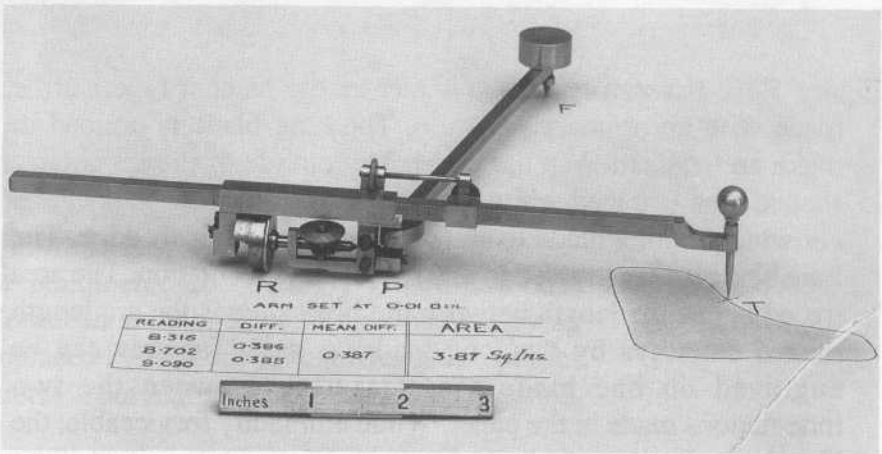


Figure 5.15. An Amsler planimeter. The arm length may be adjusted to alter the scale of units in which the record is made by the integrating wheel. Courtesy Science Museum, London. Negative No. 82.

errors, especially those arising from the motion of the integrating wheel over the unprepared paper surface. Most of these devices were superseded by the invention by Lang in 1894 of the compensating planimeter. In this form, the arms of the Amsler planimeter can be disposed in two roughly mirror-image ways when tracing an area. Averaging measurements made in these two configurations mitigate many of the errors.

A particularly simple and inexpensive form of planimeter, the Hatchet planimeter (named from its resemblance in one form to a hatchet), was developed from 1887. The principle is shown in Figure 5.16.

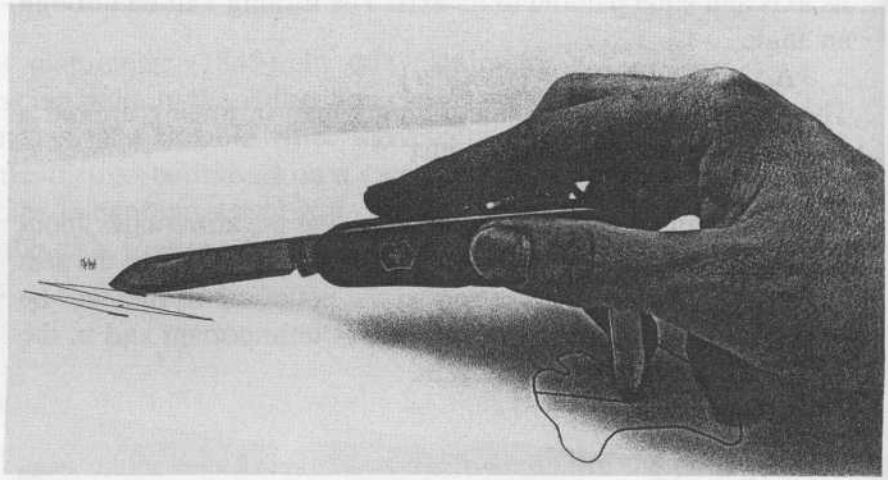


Figure 5.16. A serviceable planimeter of the hatchet type can be made with an ordinary penknife. The long blade is pressed to make an indentation in the paper. The outline of the irregularly shaped area is traced while keeping the short blade vertical and allowing the long blade to slide in the direction of its edge. The long blade is then pressed to make a second indentation. The area traced is just the length between the blades times the arc length moved sideways by the long blade. A scale for area can be engraved on one blade for measuring between the two indentations made in the paper. While eminently serviceable, the Hatchet planimeter is not a precision instrument. The mathematical analysis is complex and the result is not theoretically exact. The errors are minimized if the area is small, tracing starts near its center of gravity, and the results of clockwise and anticlockwise tracings are averaged.

An important generalization of the Amsler planimeter is the moment planimeter. An ordinary planimeter measures the area within a closed curve. By introducing additional integrator wheels that are geared so that their axes are rotated through two and three times the angle moved by the integrator arm, it is possible to measure the moment of inertia and other higher order moments of the area.[2] These moments are of considerable engineering importance, and the devices were widely used, particularly in ship design. Figure 5.17 shows a moment planimeter employing precision sphere and wheel integrators.

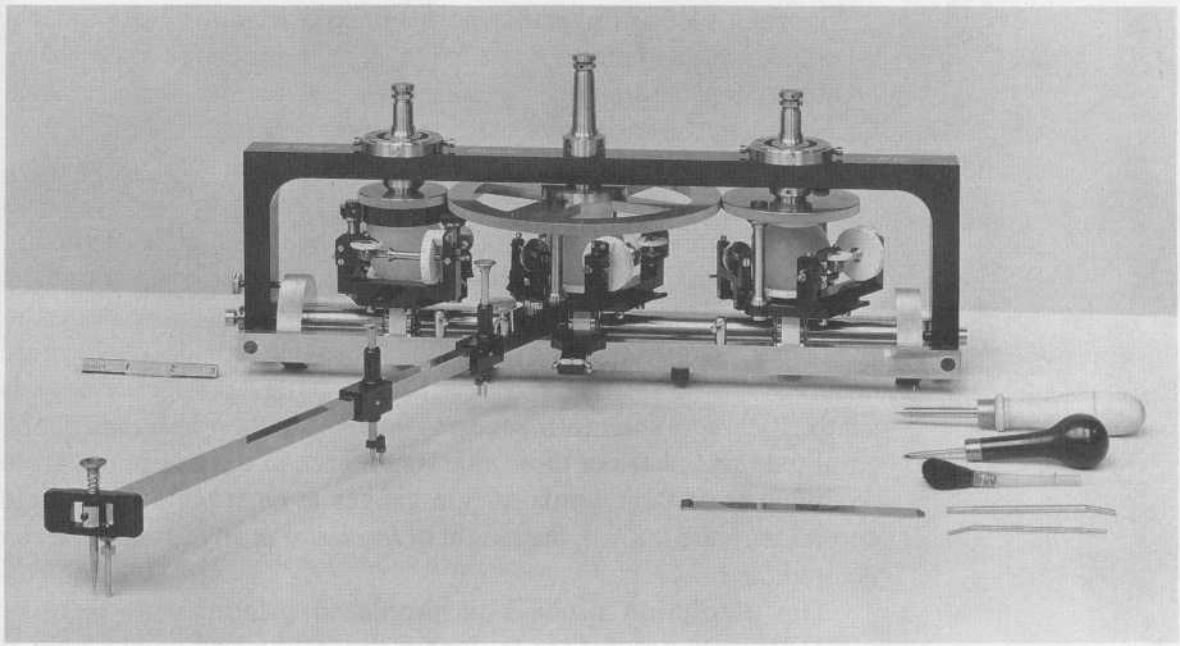


Figure 5.17. Moment planimeter designed by Hele-Shaw and manufactured by Coradi in Zurich. Integrating wheels moving over the paper are replaced by wheels moving over glass spheres for greater precision. Courtesy Science Museum, London. Negative No. 786.

The Work of Lord Kelvin

Although planimeters and their derivatives were of considerable practical importance, the importance of the mechanization of integration to other areas of mathematics was not immediately realized. It was William Thomson, later Lord Kelvin, who, in the 1870s, first grasped their wider significance. Most twentieth-century analog devices can be seen as realizations and direct developments of Kelvin's ideas.

Kelvin was led to the study of analog computing machinery from the need to predict tide heights in ports—an essential requisite to navigation in an era when dredging of channels was uncommon and in countries, such as England, where the tidal variation in water height is considerable and maritime trade was of such economic importance.

The tides are caused, primarily, by the periodic gravitational influences of the moon and the sun in their relative motions around the earth. The basic influences are diurnal, due to the earth's rotation, but additional influences of longer periods arise from such causes as the eccentricity of the earth's orbit around the sun and the inclination of the earth's axis. These basic influences are modified by the shape of the continental shelves and coastal estuaries. From the mathematical work of Fourier it was realized that the tide height could be represented by a series of sine functions of the appropriate periods for the lunar and solar influences, together with their harmonics. The amplitude and phase of these sine functions can be determined from the analysis of the records of tide gauges at each port. Once these components are known, the height of the water at any future time can be predicted.

The prediction process is simple in principle. A resolver mechanism (Figure 5.4) is set for the required amplitude and phase and driven at the appropriate rate for each component. The outputs of these are then added together, by a form of differential mechanism, to give a continuous record on a paper chart of the water height as a function of time. From this chart the times and heights of high and low water can be tabulated.

In Kelvin's *harmonic synthesizer* (Figure 5.18) the summation is performed by a wire and pulley system. Because the wire is not everywhere vertical the sine functions are slightly distorted, but the error introduced by this is small enough to be unimportant. Kelvin's *tide predictor* was completed by 1876. A second one was constructed

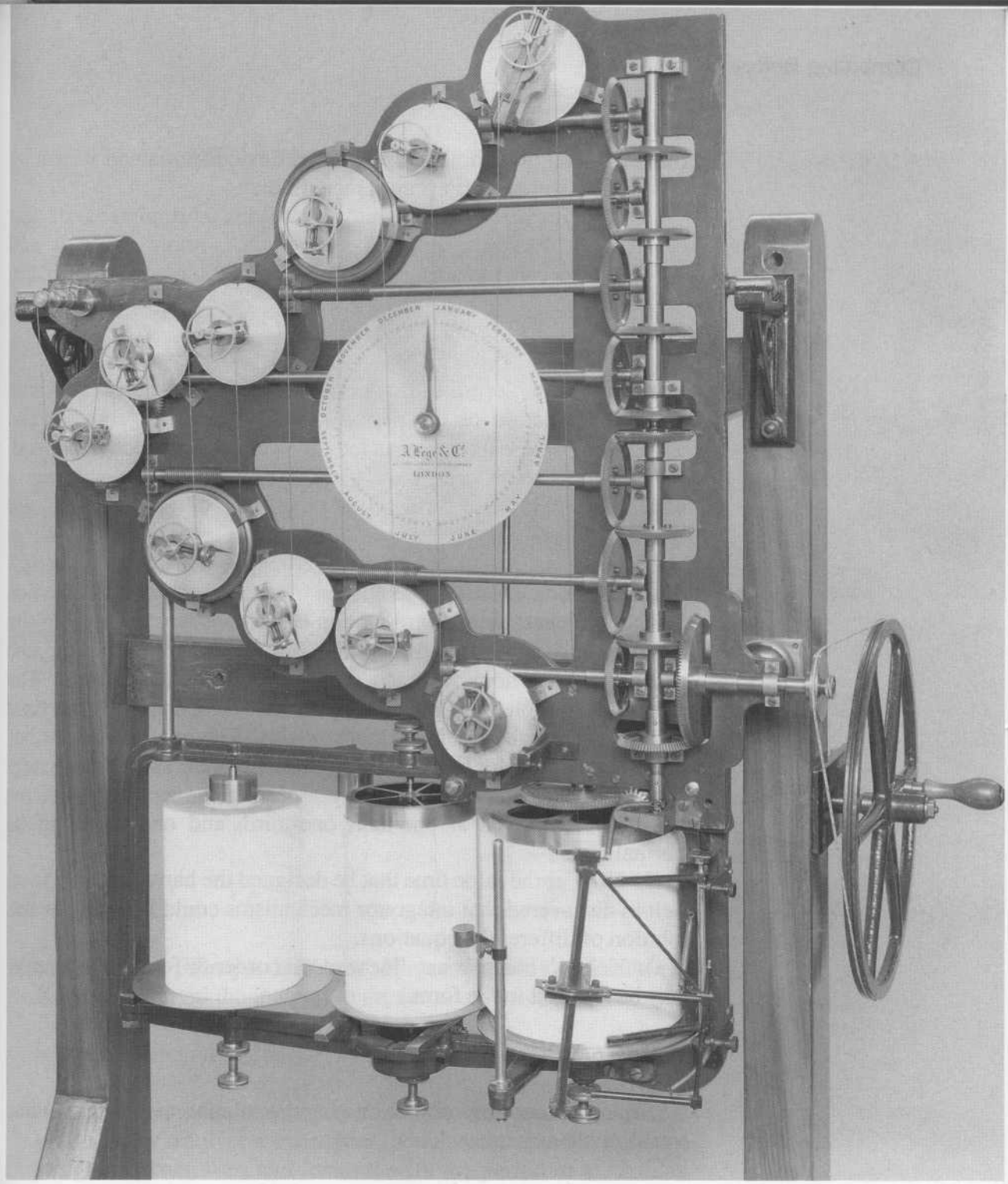


Figure 5.18. Kelvin's tide-predicting machine. Gearing from the drive handle is used to drive resolver mechanisms, which are set to generate sine functions of the required amplitude, phase, and periods. The components are added by a wire-and-pulleys system, and the resultant water height is recorded as a continuous curve on the paper roll. Tide tables are then prepared by reading the heights and times of maxima and minima of the curve. Courtesy Science Museum, London. Negative No. 86.

for the Indian government, and similar devices remained in use by all major maritime nations until recent times.

The determination of the amplitudes and phases of the components from the tide height records is more difficult and involves the evaluation, for each component, of integrals of the form

$$\int h(t)\, s(t)\, dt$$

where $h(t)$ is the record of tide height against time and $s(t)$ is a sine or cosine function of the period of the component sought. The evaluation of these integrals is a lengthy and tedious process by hand.

Kelvin used this principle as the basis of a harmonic analyzer in which he employed a form of integrator (the details of which are unimportant) devised by his brother James. The mechanism of this tide analyzer, completed in 1879, is shown in Figures 5.19 and 5.20. The integrating wheels are displaced by following the recorded tide height on a chart the forward motion of which is synchronized to the oscillation, backwards and forwards, of two discs representing sine and cosine functions for each required periodic component. The amplitude and phase of the component can be found from the final integrals shown by the integrating wheels. Kelvin's instrument has eleven integrators for five basic periodic components and the constant term. The harmonics are found by repeating the analysis with the chart record moved at one-half, one-third, and one-fourth of its normal rate.

In 1876, at the same time that he designed the harmonic analyzer, Kelvin discovered that integrator mechanisms could be used for the solution of differential equations.

In Kelvin's analysis any linear second order differential equation may be reduced to the form

$$d/dx\ (1/P(x)\ \ du/dx\ ) = u.$$

If u_i is any function of x approximating the solution to the differential equation, then

$$u_{i+1} = \int P(x)\ (C - \int u_i\, dx)\, dx$$

is a closer approximation to a solution of the differential equation. This iterative formula, Kelvin realized, had basically the same form as the products in the tidal analysis and could be carried out by two

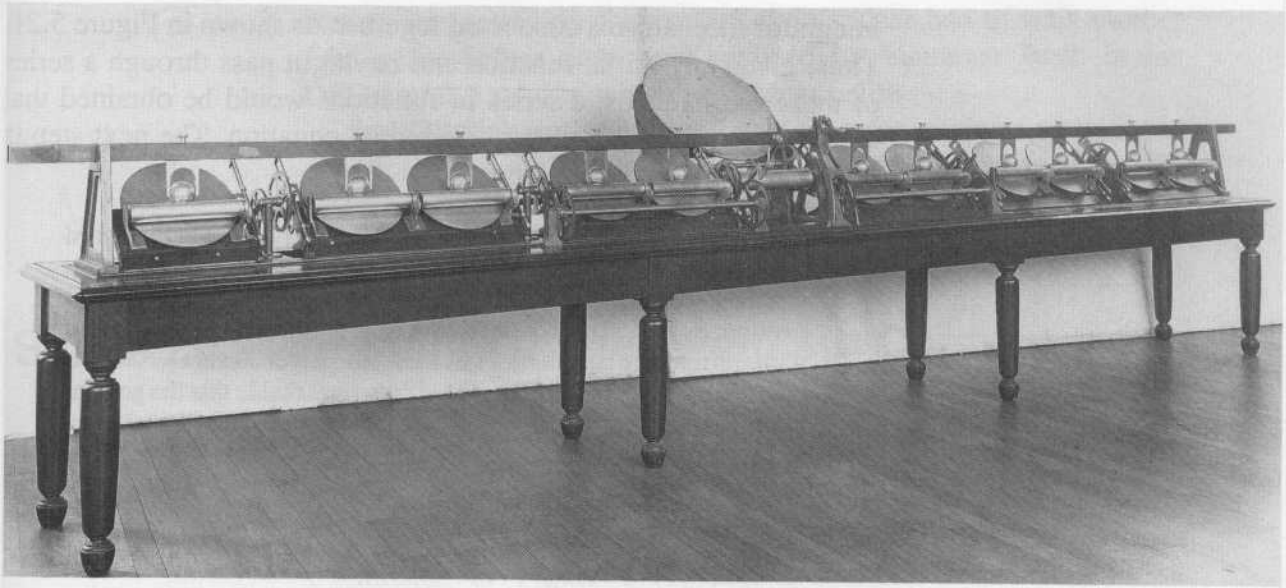


Figure 5.19. Kelvin's Harmonic Analyzer for Tides. Courtesy Science Museum, London. Negative No. 86.

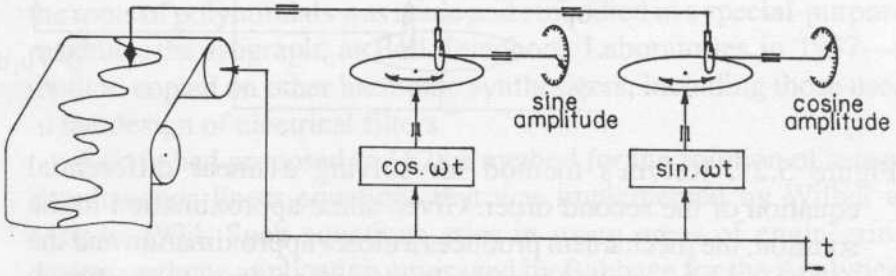


Figure 5.20. The principle of Kelvin's Harmonic Analyzer for Tides. Kelvin converted the integral to the form

$$\int h(t)\, \mathrm{d}\, (\int s(t)\, \mathrm{d}t),$$

where the integral

$$\int s(t)\, \mathrm{d}t$$

is just another sine or cosine function. These functions are easily produced by resolver mechanisms, which rotate the disks of two integrators backward and forward with the period of each tidal component sought. The integrating wheels (balls in Kelvin's design) are displaced by the recorded tide height, which is obtained by tracing a chart record. The integrating wheels then indicate the sine and cosine amplitudes, from which the amplitude and phase of the component are easily found.

integrator mechanisms connected together, as shown in Figure 5.21. Thus, given any initial function and having it pass through a series of these mechanisms, a series of functions would be obtained that converge to a solution of the differential equation. The next step is best told in Kelvin's own words:

> So far I had gone and was satisfied, feeling I had done what I wished to do for many years. But then came a pleasing surprise. Complete agreement between the function fed into the double machine and that given out by it.... The motion of each will... be necessarily a solution of [the differential equation]. Thus I was led to a conclusion which was quite unexpected; and it seems to me very remarkable that the general differential equation of the second order with variable coefficients may be rigorously, continuously, and in a single process solved by a machine.[3]

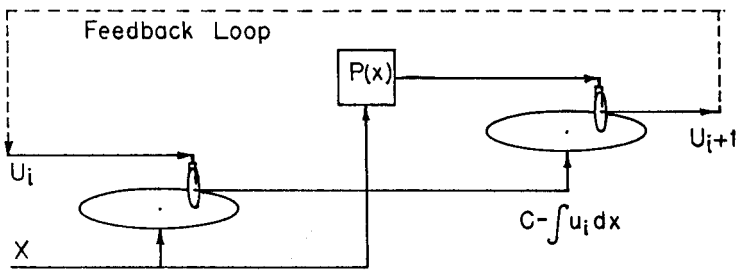


Figure 5.21. Kelvin's method for solving a linear differential equation of the second order. Given some approximation to the solution, the mechanism produces a closer approximation and the process can be applied iteratively, commencing with the newly found approximation. However, if the output and the input are connected together to form a feedback loop, the mechanism produces an exact solution to the differential equation in a single pass.

Kelvin here had discovered the basic feedback principle by which integrator mechanisms can be applied to the solution of differential equations. Although he generalized the principle to the case of any differential equation of any order, his mechanism could not be realized at the time. The basic difficulty is that the torque output from the wheel of an integrator is very slight and is inadequate to drive

further integrating mechanisms. Kelvin's ideas had to wait another fifty years before they were realized by Vannevar Bush in his Differential Analyzer, which we discuss below.

In 1878, Kelvin also invented a machine for solving simultaneous equations, essentially similar to a machine developed by Wilbur in 1934, which we will also discuss shortly.

Scientific Instruments in the Twentieth Century

Kelvin's ideas on harmonic analysis and synthesis were widely copied, and many synthesizers were built following his general plan for both tidal and general harmonic work. Coradi of Zurich manufactured a harmonic analyzer similar in style to the moment planimeter in Figure 5.17, and a number of adaptations of conventional planimeters for this purpose were also made, but no more specialized devices similar to Kelvin's analyzer appear to have been built.

An important application of the harmonic synthesizer to finding the roots of polynomials was made and embodied in a special-purpose machine, the Isograph, at Bell Telephone Laboratories in 1937—a method copied on other harmonic synthesizers, including those used in the design of electrical filters.[4]

Kelvin had proposed in 1878 a method for the solution of sets of simultaneous linear equations that was implemented by Wilbur at MIT in 1934. Such equations arise in many areas of engineering design, and one application envisaged by Babbage for the Analytical Engine had been the determination, by their means, of the orbital parameters of comets. The relative magnitudes of the variables are represented in the machine by the angles through which metal plates are turned about horizontal axes. A system of wires and pulleys is used to constrain the motions of the plates for each equation in the set, as shown in Figure 5.22. This system is repeated for each equation in the set and the plates can then only take up relative positions that represent the solution. If an approximate set of solutions is found by the machine, they may be used to find a more accurate set by an iterative process using the same machine settings except for the constant terms. The machine is, in practice, therefore capable of providing almost any desired degree of accuracy in the solutions.

Electrical technology was also used in a limited way in analog

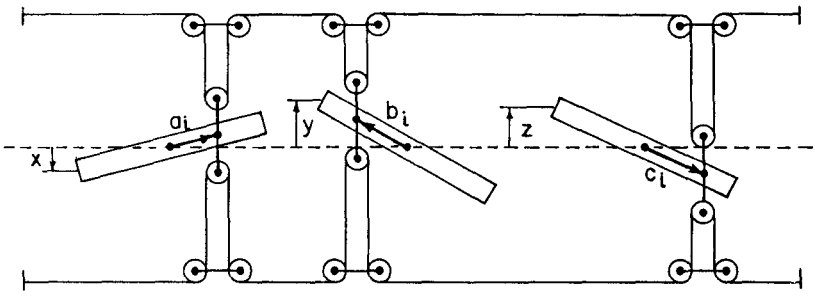


Figure 5.22. The principle of the Kelvin/Wilbur machine for solving simultaneous linear equations. The two wires running over the system of pulleys constrain the movement of the tilting plates so that $a_i\,x + b_i\,y + c_i\,z = 0$. An exactly similar arrangement is used to constrain the plates for the other equations in the set. When all the constraints are present, the relative tilts possible for the plates give a solution to the set of equations.

computing devices from World War I, but before World War II little progress had been made in abstracting these devices to the representation of mathematical functions. Rather, electrical circuits were assembled in direct analogy to the system under study—each component of the system was modeled by an electrical component that had the same functional behavior in the electrical domain as the component in the original system domain.

An important series of machines of this type were the Network Analyzers developed by General Electric and Westinghouse for the simulation of electrical power supply networks. The DC (Direct Current) Network Analyzer of 1925 used only resistive components and could, therefore, only model steady-state behavior. The AC (Alternating Current) Network Analyzer of 1929 used reactive impedances and could be used to study both phase and magnitude in alternating current power networks. Later machines could also exhibit the transient (short-term) behavior of a network in response to a surge due to equipment switching or failure. Similar electrical components and circuit techniques were used by Mallock in an electrical instrument for the solution of simultaneous equations in 1933.

Another important technique in the 1930s and 1940s was the use of electrolytic tanks, resistive papers, and elastic membranes to model continuous two-dimensional systems. In particular they were used to determine the electrical field potentials in the vicinity of the complex grids and electrodes in vacuum tubes as an aid in their design. These tubes, precursors of transistors and other modern electronic devices,

were widely used in radio and radar equipment. The height at any point of an elastic rubber membrane, for example, would represent the electrical potential at the corresponding point in the tube. Anodes, cathodes, and other electrodes would be represented by bars and rods that fixed the height of the membrane in the appropriate places. The path followed by a small steel ball rolling on the membrane would then represent the path followed by an electron in the tube. These techniques suffered the disadvantage, however, that they could not easily model the space charge effect on the potential distribution created by other electrons in motion in the space between the electrodes.

Although there were many other developments in the first half of the twentieth century akin to those we have just described, they were all of an ad hoc nature and did not lead to any general synthesis or to the emergence of a general class of machines, except for the Differential Analyzer and the Gunnery Computers to which we now turn.

The Differential Analyzer

Vannevar Bush at MIT was concerned through the late-1920s and the 1930s with the development of machines to aid the calculations with continuous functions required by design engineers. (The Wilbur machine for solving simultaneous equations was part of this work.)

Bush's first machine, an Integraph used for the integration of the product of two functions, was described in 1927. The functions are entered by following curves with pointers attached to linear potentiometers—variable resistances whose voltage or current output is proportional to the movement of the potentiometer. The integral of the product of the outputs from the two potentiometers is formed by a commercial electrical watt-hour meter. The result of this integration is followed up by a relay and servomotor system and is used to plot the integral as a continuous curve. In this way the machine can evaluate integrals of the form

$$F(x) = \int f(x)g(x)\,dx,$$

a special case of which are those same integrals for Fourier analysis

as handled by Kelvin's harmonic analyzer—ones of great practical importance in electrical engineering.

Bush realized, as Kelvin had, that by making one of the inputs follow the output (i.e., by adopting a feedback or "back coupling" principle) the machine could solve differential equations which are equivalent to

$$f(x) = \int f(x)g(x)\,dx.$$

The capabilities of the machine were extended by adding a linkage multiplier (similar to Figure 5.5) and a second integrator of the disc and wheel type (Figure 5.13) with a relay and servomotor follow-up. This machine was capable of solving most second-order differential equations of practical importance to an accuracy of 1-2%.

Bush's generalization of his integraph to the solution of a wide range of differential equations depended on the adoption of the capstan type of torque amplifier developed by Nieman for power steering in motor vehicles. The principle of this device is shown in Figure 5.23. The use of torque amplifiers meant that the small torque available from the friction wheel of a Wetli disc-and-wheel integrator

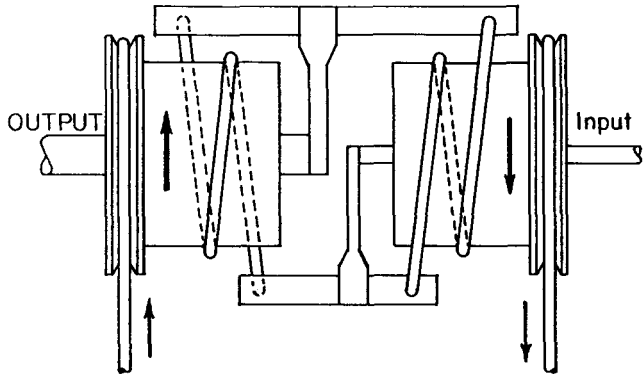


Figure 5.23. The Nieman capstan type of torque amplifier. The two drums are rotated continuously in opposite directions by an electric motor. Whenever the input shaft moves, its arm tightens the band on one drum and loosens it on the other so that the friction on the drum causes the output to be turned with the input but with a much greater torque. In the Differential Analyzer the output drives the input of another torque amplifier to give a total torque amplification of about 10,000 times.

could be used to drive a substantial load of other calculating machinery. It was the absence of any form of torque amplifier that had prevented Kelvin from making further progress in this direction.

Bush's Differential Analyzer, as the new machine was called, consists of a set of integrators, input and output tables for tracing and plotting continuous curves, and a very flexible system of shafting—the bus shafts—arranged to enable the input of any mechanism to be connected to the output of any other, as required by the problem being solved. Gearing could be included to give any prescribed ratio between bus shafts, and differential gears (Figure 5.8) enable shaft rotations to be added and subtracted. One bus shaft is driven by a motor to represent the independent variable. A typical Differential Analyzer is shown in Figure 5.24.

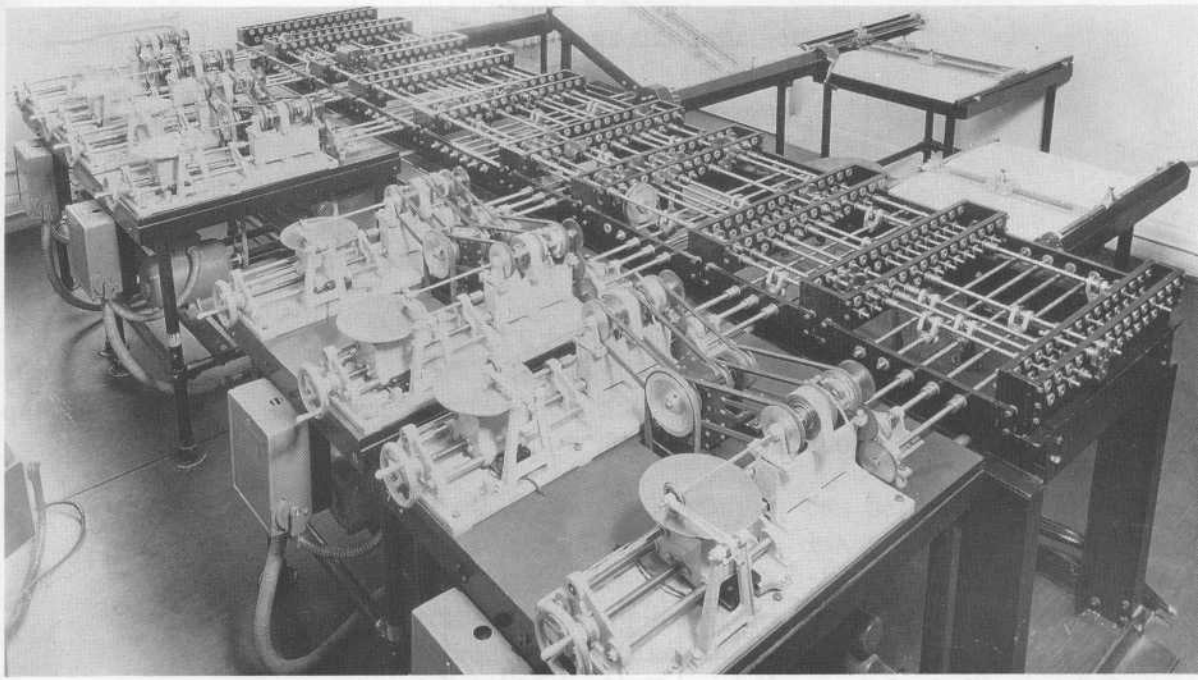


Figure 5.24. A Differential Analyzer, similar to Bush's original design, developed in the Courtaulds Laboratories. The integrators are on the left with a two-stage torque amplifier and a handle for setting the initial conditions into the integrators. On the right are the input and output tables. The bus shafting is in the center. Courtesy Science Museum, London. Negative No. 272/74.

To understand how the machine was used let us consider the solution of the differential equation for an object projected into the air in a constant gravitational field and with an air resistance proportional to its velocity. The differential equation is

$$d^2y/dt^2 + k\, dy/dt + g = 0.$$

Suppose the rotation of one bus shaft represents

$$d^2y/dt^2$$

and the independent variable shaft represents t. With one integrator we can produce an output on another bus shaft of

$$\int d^2y/dt^2\, dt = dy/dt$$

and integrating this again we produce

$$\int dy/dt\, dt = y.$$

The constant g can be represented by a shaft, appropriately preset, while the constant k is introduced by an appropriate gear ratio. Writing the differential equation in the form

$$d^2y/dt^2 = -\,[\,k\, dy/dt + g\,]$$

exhibits the feedback relationship necessary to complete the setup shown in Figure 5.25.

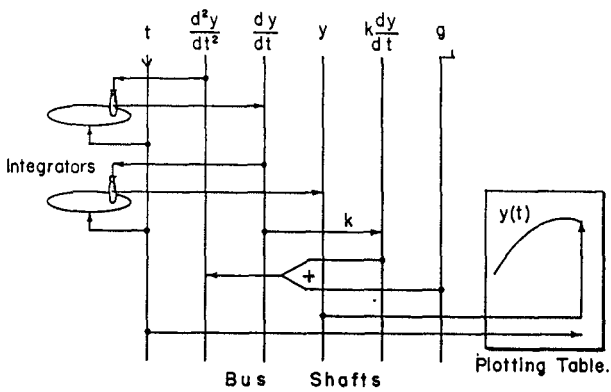


Figure 5.25. Sample setup of the Differential Analyzer for solving the differential equation
$d^2y/dt^2 + k\, dy/dt + g = 0.$

The equation just described can be solved by analytical methods. If, however, the gravitational field is a function of height, $g(y)$, and the air resistance is a general function of velocity, $f(dy/dt)$ known only empirically, then no analytical solution can be found. However, the differential equation is still readily solved on the Differential Analyzer as shown in Figure 5.26.

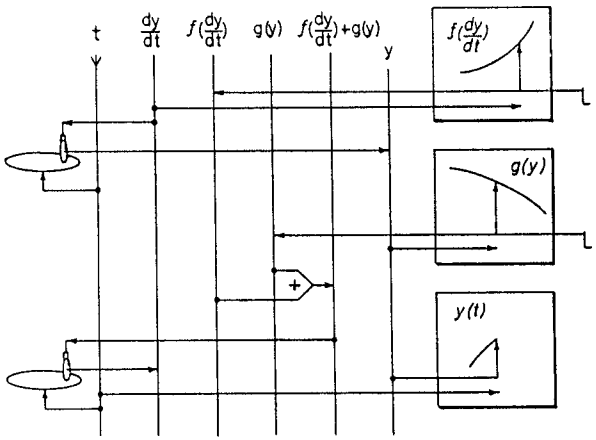


Figure 5.26. Differential Analyzer setup for solving the equation

$$d^2y/dt^2 + f(dy/dt) + g(y) = 0$$

in the form

$$dy/dt = -\int[f(dy/dt) + g(y)]\,dt,$$

with the functions f and g provided from input tables.

In that the Differential Analyzer can be set up to solve any arbitrary differential equation and this is the basic means of describing dynamic behavior in all fields of engineering and the physical sciences, it is applicable to a vast range of problems. In the 1930s, problems as diverse as atomic structure, transients in electrical networks, timetables of railway trains, and the ballistics of shells, were successfully solved. The Differential Analyzer was, without doubt, the first general-purpose computing machine for engineering and scientific use.

Bush's original Differential Analyzer provided six integrators, three input tables, an output table, and a manually operated multiplier. It could achieve an accuracy near 1 in 10^3 (0.1%). Bush's ideas were

soon copied, first by Douglas Hartree in Manchester. Hartree made a demonstration model using mainly components from the Meccano construction toy system, which yielded an accuracy of about 1 in 10^2 (1%) and proved surprisingly useful in calculations of atomic structure, before embarking on the construction of a large-scale and more accurate machine. In total, about nine major Differential Analyzers were in operation before World War II, and at least as many again were constructed during the 1940s.

Much of the art in using the Differential Analyzer lay in molding the equations to suit the forms available on the machine. This led, in particular, to demands for additional integrators for a variety of auxiliary purposes. In the setup of Figure 5.25, for example, the constant k is introduced as a gear ratio. If it were desired to investigate the dependence of the solution on this constant, it would be necessary to change the gear ratio before each run of the machine. A much more convenient approach would be to set the constant k on a bus shaft and to use an integrator as a constant ratio drive that is simply varied between runs.

A more profound application arises when multiplication occurs. Instead of using a special multiplying mechanism, two integrators, connected as suggested by the formula

$$uv = \int u dv + \int v du,$$

can often be employed. It was a great strength of the Differential Analyzer, not found in later electronic analog computers, that integration could be performed with respect to any variable represented by a shaft in the setup.

In a similar manner, a sine or cosine function, perhaps for use as a forcing function when studying the behavior of a car suspension on a rough road, can be introduced by solving the auxiliary differential equation

$$d^2z/dt^2 = -kz$$

as part of the setup. Much ingenuity was expended in finding simple and economical setups for a wide variety of functions occurring in differential equations.

Hartree successfully extended the use of the Differential Analyzer to problems involving a time-delayed function (by having an input pointer trace an output curve somewhat behind the output pen) and,

with more limited success, to the solution of some partial differential equations arising in heat flow and similar problems.

Bush had the last word in performance of Differential Analyzers when, in 1942, a new Differential Analyzer was produced at MIT. In this machine an accuracy of 1 in 10^5 in the components was sought to achieve better than 1 in 10^4 (0.01%) accuracy in the solution of differential equations—about ten times greater accuracy than possible with any other machine. Variables were transmitted in this system not by shaft rotations but by electrical signals derived from capacitative encoders on the integrating wheels, which were reconstituted to mechanical motions by servomotors as required. The interconnection of the components was determined by a system of relays, themselves controlled by information read from punched tapes. The setting of initial conditions in the integrators, etc., was also controlled by punched tape so that no manual actions were required to set a problem into the machine. It was even possible to run separate problems simultaneously in different parts of the machine. The setup task which had previously taken hours or days now required only minutes, for the preparation of the tapes could be carried out away from the machine. In this way, the throughput of the Differential Analyzer was greatly increased.

Despite the sophistication of Bush's second Differential Analyzer, simpler, fully mechanical machines similar to his first design continued to be made into the 1950s because of their much lower cost. Differential Analyzers were finally superseded by electrical analog computers of generally lower precision, and later by digital computers, because the precision mechanical work required in the construction of a Differential Analyzer made them prohibitively expensive. But the mathematical flexibility of the Differential Analyzer was never matched by electrical analog computers and originally only with difficulty by digital computers. Figure 5.27 shows an ingenious postwar development in which mechanical components are interconnected by flexible steel tapes rather than bus shafts.

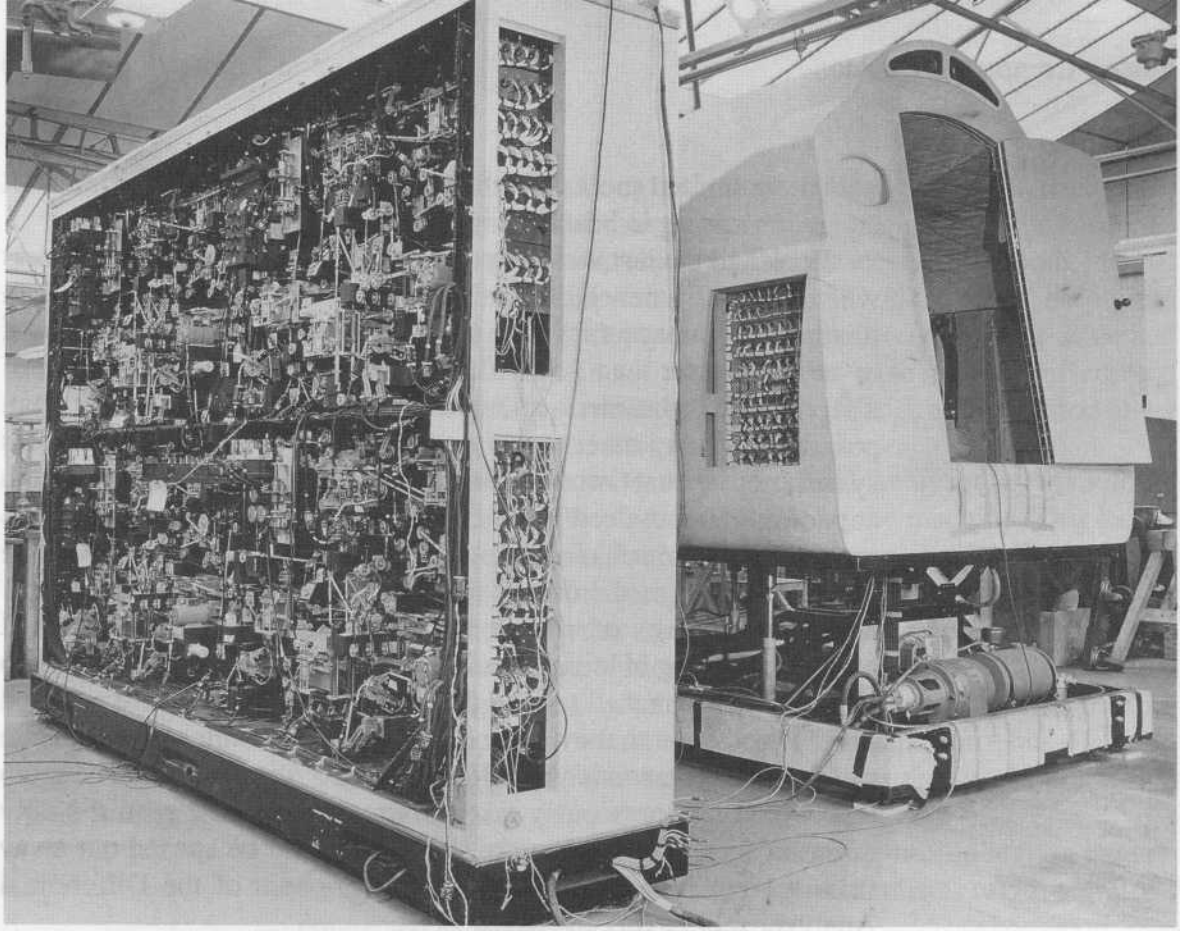


Figure 5.27. An experimental flight simulator for the Viscount aircraft (ca. 1950). Mechanical analog computing components are interconnected by flexible steel tapes running over pulleys to combine the accuracy of mechanical computation with the flexibility of interconnection of electrical analog systems. Courtesy Science Museum, London. Negative No. 306/73.

Gunnery Computers

Differential Analyzers have considerable historical importance as the first general-purpose automatic computing devices for scientific and engineering work. However, because of their cost, they were never very common nor their use widespread.

In terms of practical use, mechanical analog computing devices played their most dominant role in military applications, particularly

for the aiming of guns and other weapons from moving platforms or at moving targets. Although inherently special-purpose in nature, gunnery computers reached a high degree of sophistication because of the mathematical complexity of gunnery problems and the urgent military need. Most importantly, the need to solve these problems continuously in real time, with a delay no greater than a small fraction of a second, made other forms of calculation of little use and provided a secure niche for analog computers until well into the 1970s.

Military analog computers had their origin before World War I in naval gunnery to control the aiming of guns against moving targets. Because the speed of ships is small compared with the velocity of shells, simple linear approximations generally suffice, and the main task of the mechanism is to continuously keep track of the range and bearing of the target. Such computing mechanisms were developed in England by Pollen and Dreyer. In America, Hannibal Ford introduced in his gunnery computers an integrator employing two balls squeezed between an integrator disc and an output cylinder. Because no sliding movement is required in its operation the components of this integrator can be squeezed together with substantial pressure, so the torque output is much greater than that of a Wetli disc-and-wheel integrator and is adequate for most purposes without torque amplification. Although the accuracy of the disc-and-ball integrator is not high, it formed the basis of most military applications because of its simplicity, robustness, and convenience.

Rapid development of gunnery computers occurred in the 1920s in response to the substantial military threat from aircraft, as demonstrated in the final years of World War I. In essence, the antiaircraft gunnery problem is straightforward. The aircraft is tracked with instruments to determine its present position. From a continuous series of observations the course and speed of the aircraft can then be found. The course is extrapolated for the time of flight of the shell, assuming the course and speed to remain constant, to give the aircraft's future position when the shell reaches it. A feedback loop is involved because the time of flight of the shell depends on the future position of the aircraft.

The problem is made difficult by practical considerations. The shell must be aimed, and timed by a fuse mechanism, to explode within about 30 feet (10 meters) of the aircraft, which, if flying at high altitude, might travel 1 mile (1.6 kilometers) or more during the flight of the shell. No very effective range-finding instruments were available until the introduction of radar at the start of World War II.

The time to locate the aircraft, set up the calculation, and start firing the guns was short—about thirty seconds—and thereafter the computer had to continuously provide updated firing data to the guns.

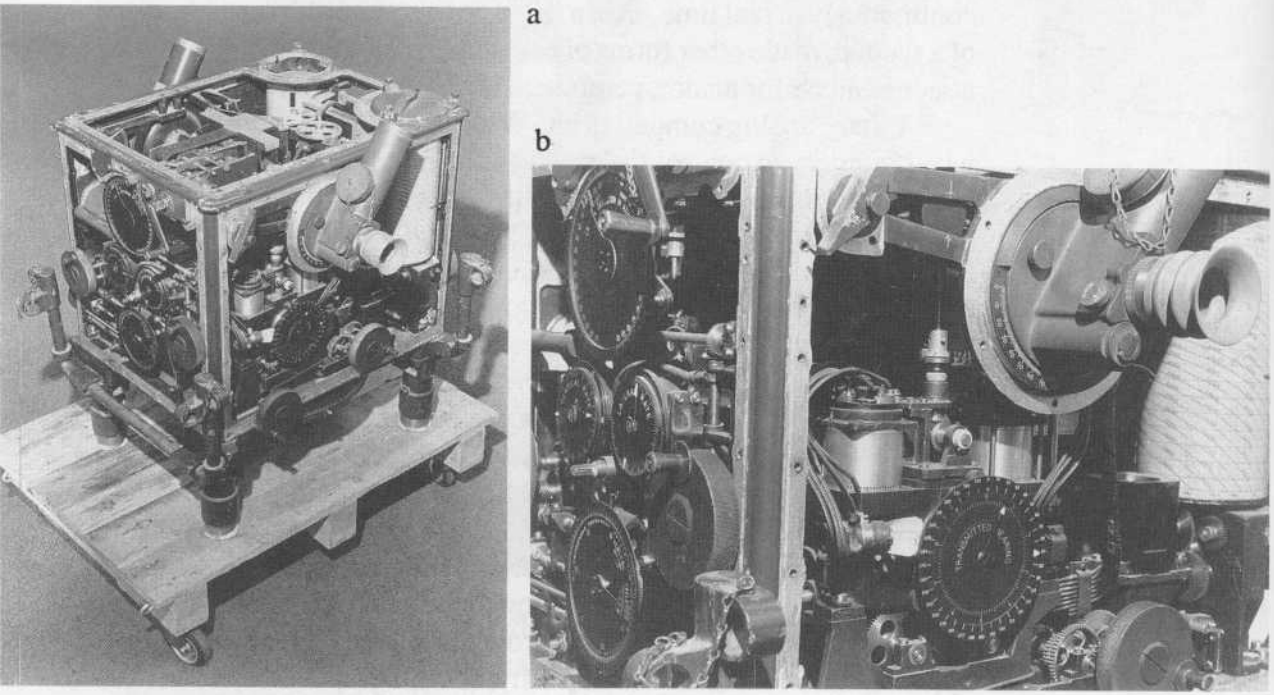


Figure 5.28. (a) and (b) The Vickers antiaircraft gun Predictor (ca. 1930).

The first successful antiaircraft gun computer, the Vickers Predictor (Figure 5.28), was developed by the English armament manufacturing firm of Vickers in 1924 and entered service in 1928, well before the invention of the Differential Analyzer. The general arrangement of the mechanism, which uses polar coordinates to achieve the necessary accuracy, is shown in Figure 5.29. Linkage mechanisms (like the resolver of Figure 5.4) were extensively used in the design, and disc-and-ball integrators drove the balance dials. Operators were required to enter deflections to keep the dials, and hence the equations, balanced and to enter ballistic data by following curves on drum charts. These operators acted, in effect, as servomechanisms, and the action of the predictor was entirely

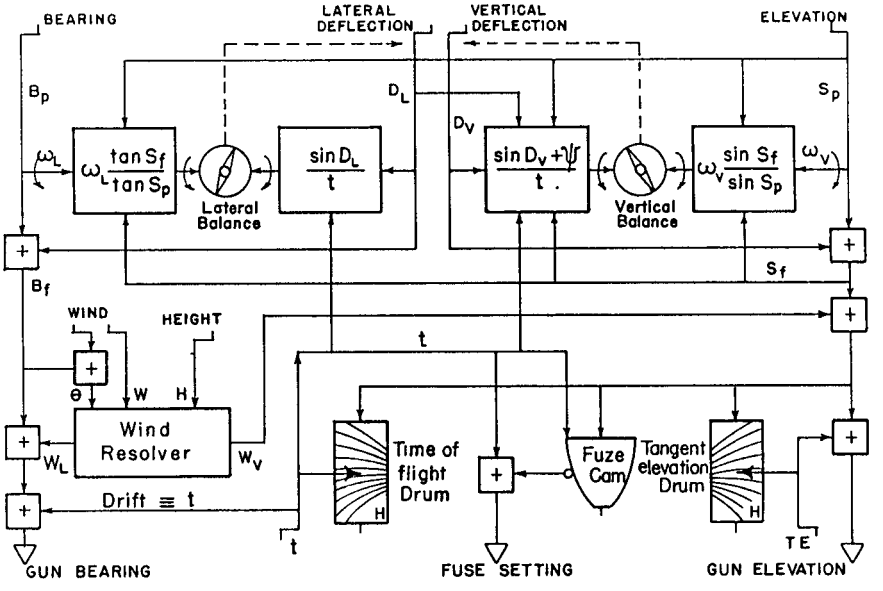


Figure 5.29. General arrangement of calculation in the Vickers antiaircraft gun, Predictor. The calculations were done in polar coordinates to achieve the required accuracy, but the basic equations the Predictor was required to solve were then quite complex:

$$\sin D_L/t = \omega_L [\tan S_f / \tan S_p]$$

and

$$[\sin D_V + \psi]/t = \omega_V [\sin S_f / \sin S_p],$$

where

$$\psi = (1 - \cos D_L) \sin S_p \cos S_f.$$

mechanical, even to the extent of employing a clockwork gramophone motor to drive the integrator discs. An antiaircraft gun predictor employing Cartesian coordinates was introduced by Sperry Gyroscope Co. in the early 1930s, and similar devices were developed for naval purposes. Some five to ten thousand such instruments were used in World War II.

The evolution of gunnery computers seems to have been completely independent of the evolution of civilian Differential Analyzers until near the outbreak of World War II. During the war the Differential Analyzers were taken over for military purposes,

principally for the solution of the differential equations of shell trajectories in the preparation of ballistic firing tables. Development of gunnery computers in World War II rapidly responded to the greatly improved capabilities of aircraft. This involved the use of much more sophisticated mathematical principles in the gunnery computers, such as the autobalance mechanism sketched in Figure 5.30, and the extensive use of servomechanisms to reduce the requirements for manual operators and increase the accuracy of the results. These developments were influenced by the mathematical techniques used in designing setups for Differential Analyzers and in turn influenced further such developments after the war.

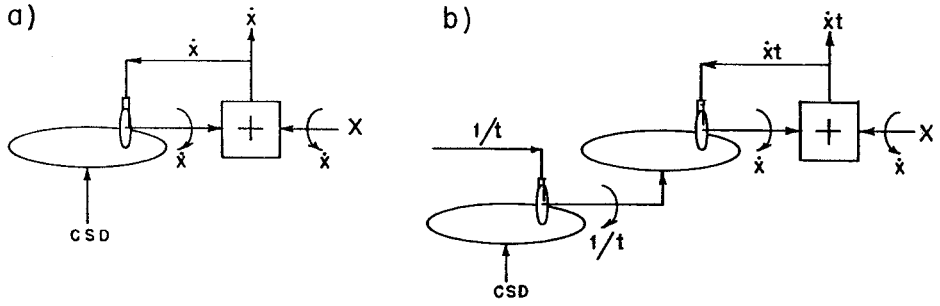


Figure 5.30. The principle of the auto-balance mechanism employed in the Sperry Predictor. In diagram (a) is shown the interconnection of an integrator to act as a differentiator. The input is a shaft position representing a position coordinate, x, of the aircraft. This shaft turns at a rate x as shown by the curved arrow annotation. If the integrating wheel is displaced from the center of the disk driven from a constant speed drive (CSD) by an amount x, then the integrating wheel will rotate at a rate x. The input and the output from the integrating wheel are subtracted in a differential. If the two rates are not equal, the output of the differential will move the integrating wheel across the disk to restore the balance. This mechanism does not respond instantaneously to a change in the input rate but approaches it exponentially with a time lag dependent on the gear ratios and other constants of the mechanism. By driving the disk at a rate $1/t$, where t is the time of flight of the shell, the output is made xt as shown in the diagram (b).

Many other types of mechanical analog computing mechanisms were developed for military uses in World War II. Most common were the bomb sight computers used in aircraft for aiming at ground and ship targets, and the computers for directing defensive guns on bombers against attacking fighter aircraft. The linkage computing mechanisms, of very simple construction but very sophisticated design, developed by Svoboda at MIT (described above) deserve particular mention.

After World War II, designs of mechanical gunnery computers were greatly elaborated and their use expanded to many military applications. They continued to be developed into the 1960s and beyond, and remained in active military service well into the 1970s. Some, such as the 1943 mechanical analog computers directing the heavy guns of the USS *Ohio*-class battleships, are still in active service. Later devices were frequently a hybrid of mechanical and electrical analog devices, particularly in avionics applications.

Analog computers yielded only slowly to electronic digital devices. The exact pattern is difficult to follow because of security restrictions, but until recent times the major applications of digital computers appear restricted to command post and other tactical control systems, rather than direct weapon control. The major reason for the demise of analog systems was the gradual replacement of guns by missiles and other self-guided weapons, for which no accurate aiming system was necessary, although the missiles themselves frequently contained analog guidance systems.

Thus, although mechanical analog computing devices played only a small, but important, role in scientific developments in the twentieth century, they played a dominant role in military computing for fifty to sixty years and were very extensively applied.

Electrical Analog Computers

World War II produced remarkable technological advances in many areas of human endeavor but in no area were the consequences so profound as in electronics. In analog computation the war resulted in the emergence of an electrical computing technology to rival the earlier mechanical technology.

This was stimulated in no small part by the scarcity of the skilled labor required to manufacture and maintain precision mechanical

systems. The military, for example, was largely satisfied with existing mechanical gunnery computers. It was not until near the end of the war that electronic or hybrid analog devices began to offer functional advantages over their mechanical predecessors, with the replacement of human operators by more reliable servomechanisms and the gradual direct coupling of inputs from radar systems.

One single project, the development of the M-9 antiaircraft gunnery computer by the Bell Telephone Laboratories, shaped the future of electrical analog technology in a way even more significant than the ENIAC did for digital technology.

The basic computing elements of the M-9 were nonlinear potentiometers made by winding fine resistance wire on nonlinear formers. With these the output voltage is not directly proportional to the position of the input but can be made proportional to geometric or ballistic functions occurring in the gunnery problem. By making the height of the former proportional to the derivative of the desired function any reasonable monotonic function can be generated. This idea can be traced back to experimental antiaircraft gunnery computers in the last years of World War I. The principle was rediscovered in the Bell Labs about 1940.

The use of potentiometers as computing elements suffers one great fault that had made previous devices unsuccessful. If the output of the potentiometer drives other circuitry, the current drawn distorts the function generated by the potentiometer so that the accuracy of the computation is seriously compromised.

This difficulty was overcome in the M-9 by having each potentiometer drive an amplifier with high-input impedance to isolate the output of the potentiometer from the input of succeeding circuits. These amplifiers were of high gain (about 10,000 power) but connected in a feedback arrangement so that the output of the amplifier was continuously compared with its input. In this way the output was made insensitive to any fluctuations in the gain of the amplifier. This arrangement is called an "operational amplifier" and was developed by Lovell at Bell for the M-9. The same idea had been independently discovered by Philbrick in 1938. A similar feedback loop was used to control the servomotors that drive the inputs of the potentiometers.

An operational amplifier can be used to perform a wide range of computing functions by suitably arranging the feedback circuit. Some simple arrangements typical of those used in the M-9 are shown in Figure 5.31. After the war operational amplifiers became the basis of

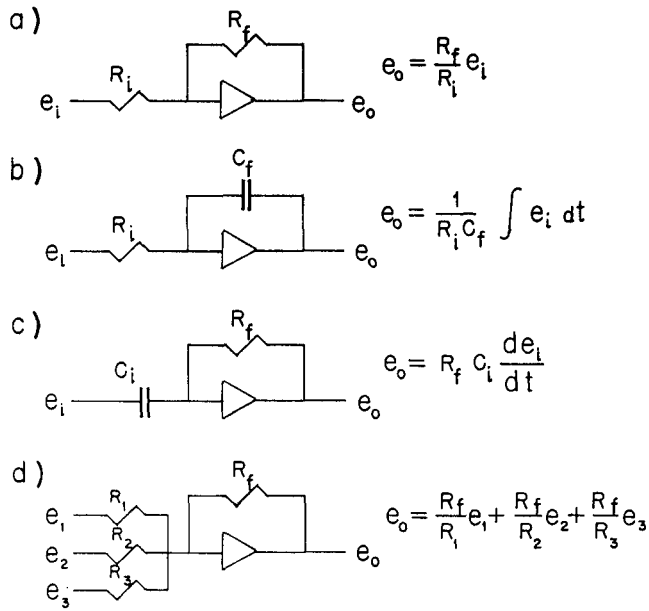


Figure 5.31. Simple feedback circuits employing an operational amplifier as (a) and isolating amplifier and scale changer; (b) an integrator; (c) a differentiator; and (d) an adder. Differentiation is normally avoided because of its sensitivity to noise in the input. More elaborate functions can be obtained with more complex feedback circuits.

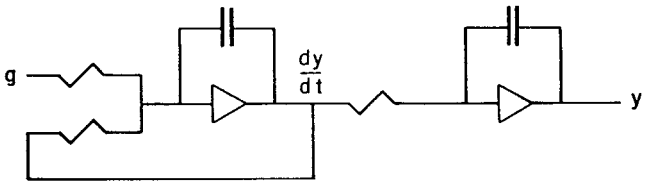
electrical analog computers and were interconnected, as shown in Figure 5.32, to solve differential equations in an analogous manner to the setup of Differential Analyzers. However, although Differential Analyzers required a high-precision shafting system for interconnection and high-precision mechanical components, the operational amplifiers could be simply and inexpensively interconnected by wiring. Because of their low cost, and despite their generally lower accuracy, electrical analog computers became enormously popular in the 1950s for solving the wide range of engineering and scientific problems that were suitable for the Differential Analyzer. One logical deficiency of electrical analog computers that needs mention is the fact that integration and differentiation can only be performed with respect to time, so that many of the techniques for setting up equations on the Differential Analyzer are inapplicable.

Figure 5.32. An electrical analog computer setup, analogous to the Differential Analyzer setups of Figs. 5.25 and 5.26 for solving the differential equation

$$d^2y/dt^2 + k\,dy/dt + g = 0$$

in the form

$$dy/dt = -\int[k\,dy/dt + g]\,dt.$$



As in the M-9, electrical analog computers used potentiometers for generating complex functions. These potentiometers were moved by servomotors directed by electronic amplifiers. Multiplication of two variables is a typical function normally performed in this way. Much ingenuity was expended on ways to utilize electrical analog computers most effectively.[5]

Although the direct current type of electrical analog computer just described became the standard type for laboratory use in the 1950s and 1960s, it had serious competition from alternating current devices in military and aviation applications. AC has some technical advantages. For example, trigonometric functions can be readily generated by the inductive coupling of one coil turned at an angle to another as shown in Figure 5.33.

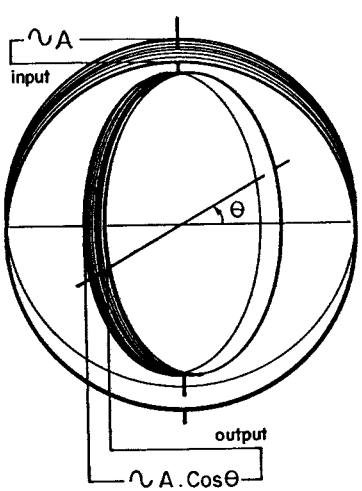


Figure 5.33. The principle of the AC electrical resolver.

Motorlike devices well suited to this purpose had been widely developed during World War II for the remote transmission of shaft angles under the name of Selsyns or Magslips. Analog computer systems assembled from such components were relatively small and light and were extensively used in the 1950s and 1960s in military and civilian aircraft systems. Frequently they were hybridized with some mechanical analog computing components. As noted above, many such systems remained in use well into the 1970s.

Conclusion

There are two fundamentally different ways in which a numerical value can be represented in a calculating device. In an analog device a direct proportion is established between the quantity represented and the position of a sliding or rotating part in a mechanical system or the voltage in an electrical circuit. Conceptually, the mechanism can take on any value in a given range. In a digital device each part can take on one of only a finite set of states, and a group of similar devices is necessary to represent a number as a string of digits in some number system.

Any mechanical or electrical device is, in practice, subject to disturbing influences, conveniently called "noise." If a digital device is disturbed by a small amount, no error in its indication will arise. In a mechanical device a spring detent will restore a wheel to its correct position. In an electronic circuit an amplifier, such as is present in the output of any logic gate, will restore any disturbed voltage to defined and well-separated discrete ranges. Digital systems can, in practice, be made immune to the effects of noise and a number can be represented to any arbitrarily high precision by a large enough group of similar digital devices.

An analog device, in contrast, has no noise immunity. If its state is disturbed, the new state represents a perfectly valid value of the variable, and the mechanism can in no way distinguish the new state from the original. The precision that can be achieved in an analog computer is, therefore, entirely limited by how small the noise can be kept. High precision, whether in the machining of mechanical parts, the manufacture of electronic components, or the isolation of

a circuit from external electromagnetic influences, is always difficult and expensive to obtain. Therefore, although a precision near 1 part in 10^3 (0.1%) is, with care, readily enough obtained in an analog device, 1 in 10^4 (0.01%) is difficult and expensive, and 1 in 10^5 (0.001%) is rarely obtained. As an example, a simple 25 centimeter, straight slide rule yields a precision of about 1 in 300, but to achieve 1 in 10^4 a scale length of about 10 meters and an elaborate construction, such as the Fullers helical slide rule or the Thatchers grid iron slide rule, is necessary.

Counterbalancing the limited precision of analog devices is their general simplicity of form and consequent economy of manufacture. In a digital device, a large group of parts is needed to represent a number. Only addition of numbers is commonly available as a digital function. To calculate a sine, for example, we must usually evaluate a power series—with the multiplications this implies—carried out, in effect, by a series of additions. Evaluation of a sine function in a digital system is, therefore, both complex and relatively very slow compared with an analog system (Figures 5.4 and 5.33).

In practice, two further characteristics distinguish analog and digital devices, although a few exceptions can be found. Digital devices, such as a mechanical calculator or an electronic digital computer, are usually general-purpose devices that can readily be adapted to carry out a wide range of tasks. If a digital device can calculate a sine function, for example, it is usually easy to adapt it to calculate a logarithm or Bessel function. An analog mechanism for the sine function, however, would be no use at all for a logarithm.

For this reason, analog devices are normally composed of a number of independent parts each designed to perform a single, distinct part of a calculation. Because of their independence these parts can all act simultaneously in parallel with one another. Analog devices are, therefore, well suited to real-time applications in which the calculation must keep step with events in the external world. In contrast, digital systems typically reuse a single calculating unit sequentially for each step in a calculation, so that the speed is reduced as the complexity of the calculation increases.

The earliest calculating aids, the counting table and abacus, were digital, as they demanded very little in manufacturing skill yet combined a generous degree of precision with simple and effective use. Napier's invention of logarithms reduced multiplication to the simpler operation of addition and made possible the slide rule, a simple yet very effective analog device in which only a limited

precision was required. Mechanical digital calculating devices first became widely available late in the nineteenth century when improved manufacturing techniques made the multitude of components required available at an economical price. Their limited speed of action effectively restricted these calculators to commercial calculations involving only simple addition operations.

Through the first half of the twentieth century, mechanical analog devices were developed extensively, particularly to handle the real-time calculations required in military applications. World War II brought the electronic technology that first made it possible for digital devices to operate at sufficient speed to perform higher mathematical functions than addition, at a speed competitive with even slow mechanical analog devices. The same technology also made possible improved analog devices that competed effectively with digital devices in many areas until well through the 1960s. It was the remarkable cost reductions of electronic digital devices in the 1970s that finally enabled them to supplant analog devices as the dominant technology for calculation.

Notes

1. In that integration is fundamentally a concept of the calculus, we must have some recourse to higher mathematics in the remainder of this chapter. Where possible, however, we have confined this to notes and figure captions. The other fundamental concept of the calculus is differentiation. This has very little application in mechanical devices—while an integrator tends to average out any small random errors in its input, a differentiator tends to accentuate them.

2. By setting the additional integrating wheels at angles of
   $\pi/2 - 2\alpha$ and 3α,
   the moments
   $y^2\,dx$ and $y^3\,dx$
   can be found by using the trigonometric identities
   $\sin^2\alpha = 1/2 - 1/2(\cos 2\alpha)$
   $\sin^3\alpha = 3/4 \sin\alpha - 1/4 \sin 3\alpha$.

3. Thompson W. [Lord Kelvin], *Treatise on Natural Philosophy, Vol. 1* (Cambridge: Cambridge University Press, 1890), 498.

4. The method is based on De Moivre's theorem for complex numbers. If

$$z = r(\cos\theta + j\sin n\theta)$$

then

$$z^n = r^n(\cos n\theta + j \sin n\theta).$$

Given a polynomial

$$f(z) = a_n z^n + \ldots + a_1 z + a_o$$

we then have

$$f(z) = (a_n r^n \cos n\theta + \ldots + a_1 r \cos\theta + a_o + j\,(a_n r^n \sin n\theta + \ldots + a_1 r \sin\theta).$$

Both the real and imaginary parts are easily formed by a harmonic synthesizer for any given value of r. If both parts are plotted simultaneously in the complex plane, the result is a closed curve that circles the origin exactly as many times as there are roots of the polynomial with their modulus less than r. The roots can, therefore, be found by systematically varying r and replotting.

5. For example, if a circuit is available to form the square of a variable, then multiplication can be reduced to addition by taking advantage of the quarter-squares formula

$$ab = 1/4\,[(a + b)^2 - (a - b)^2].$$

Further Reading

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