SECTION 2 - ANALYSIS OF OPERATION

A. SQUARE ROOT MATHEMATICS

The EC-130 uses a modified "Sum of the Odd Integers" method of extracting a square root. The method usually taught in school is not easily adapted to digital techniques, thus another system must be used. The sum of the odd integers is but one of many ways of arriving at a square root.

Odd integers are the whole numbers 1, 3, 5, 7, 9, 11, etc. If such a succession of odd integers, (odd-order arithmetic progression), are added, each summation will result in a perfect square. A perfect square is a number whose square root is a whole number. The following table illustrates this.

1) $1 + 0 = 1 = 1^2$
2) $1 + 3 = 4 = 2^2$
3) $1 + 3 + 5 = 9 = 3^2$
4) $1 + 3 + 5 + 7 = 16 = 4^2$
5) $1 + 3 + 5 + 7 + 9 = 25 = 5^2$

By adding the next higher odd order integer to each preceding summation, all existing perfect squares are produced, and none is omitted. Note that the NUMBER of odd order integers in each summation IS the square root of the sum. In example 5, there are 5 parts to the summation, and 5 is the square root of 25. This holds true for any combination of successive odd integers.

As another example:

$1 + 3 + 5 + 7 + 9 + 11 + 13 = 49$
1; 2; 3; 4; 5; 6; 7;

Therefore, $\sqrt[2]{49} = 7$

By reversing this procedure, the square root can be extracted. In this case, we SUBTRACT successive odd integers to arrive at the root, starting at 1.

For example:

16	15	12	7
- 1	- 3	- 5	-7
15	12	7	0
(1)	(2)	(3)	(4)

Therefore, $\sqrt[2]{16} = 4$

The number of successful subtractions is the square root. If a larger number is used, more subtractions are necessary.

64	63	60	55	48	39	28	15
- 1	- 3	- 5	- 7	- 9	-11	-13	-15
63	60	55	48	39	28	15	0
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)

The square root of 64, then, is 8.

This is a satisfactory method for use with perfect squares. A slight change is necessary, however, to alter the method for numbers that do not have an integral square root. Rather than count the subtractions to arrive at the root, we can apply a simple formula.

In the last example, ($\sqrt[2]{64}$), note that the last number successfully subtracted is 15. Adding 1 to this, ($15 + 1 = 16$), then dividing by 2, ($16/2 = 8$), gives the square root in another way. Expressed as a formula:

$R = (n + 1)/2$, when R is the last integer to be successfully subtracted, (no overdraft), and is sometimes called the "partial root".

An example will show the complete method developed so far.

100	99	96	91	84	75	64	51	36	19
- 1	- 3	- 5	- 7	- 9	-11	-13	-15	-17	-19
99	96	91	84	75	64	51	36	19	0

$$R = \frac{n + 1}{2} = \frac{19 + 1}{2} = \frac{20}{2} = 10$$

Thus, the square root of 100 is 10.

The method developed to this point is perfectly valid for small numbers, but it is cumbersome for large numbers. Since the number of subtract cycles is equal to the square root of the number, an excessive length of time is necessary to complete a problem using large numbers. To avoid this, another change must be made.

Starting at the decimal point, the number is divided into pairs of digits, called couplets. Beginning at the most significant couplet, the answer can be derived, one digit being produced for each couplet. However, the couplets must be operated upon in a way that is somewhat different than was done in the previous examples.

If the number that is to have its square root extracted is 441, it appears as 04 41 when divided into couplets. The root of each couplet will become one digit of the final square root, which will have 2 digits to the left of the decimal point. However, the root of 04, and the root of 41 cannot be combined in any way to produce the root of the original number. The following example will serve to illustrate the method used with the couplets.

The first step is to operate upon the most significant couplet by subtracting successive odd integers.

```
 04     03
+ 1    - 3
---    ---
 03     00
```

Now, we do not apply the formula in its entirity. A 1 is added to the 3, but we do not divide by 2 yet. 3 + 1 = 4, and this is used to begin operation on the second couplet. First, however, since the partial root for the most significant couplet is in the 10's column of the final result it must be altered slightly by multiplying by 10.

The second step is to use this result, (4 x 10 = 40), add 1, and begin operations on the second couplet.

```
 41
-41
---
 00
```

Now we can apply the formula, and (41 + 1)/2 = 42/2 = 21

Thus, the square root of 441 is 21. In this example, there is no remainder, so the problem is completed at this point.

A number that results in a remainder in the first step is treated in much the same manner. For example, let us extract the square root of 841.

```
Step 1.          08   41  (couplets)
                -01
                ---
                 07

                 07
                -03
                ---
                 04

                 04
                 05
                ---
            (overdraft)
             (restore)
```

In this case, 3 is the largest odd integer successfully subtracted and so becomes the partial root. Multiplying by 10 to adjust for the columnar position, after incrementing by 1, results in (3 +1) x 10 = 4 x 10 = 40.

Step 2. The remainder above, (04), combines with the other couplet (04 41). Use the partial root, (40), again incremented by one as we repeat the odd integer method.

04 41	04 00	03 57	03 12	02 65
- 41	- 43	- 45	- 47	- 49
04 00	03 57	03 12	02 65	02 16

02 16	01 65	01 12	00 57
- 51	- 53	- 55	00 57
01 65	01 12	00 57	00 00

The last number successfully subtracted is 57. Again applying the formula:

$$R = (n + 1)/2 = (57 + 1)/2 = 58/2 = 29$$

Thus, the square root of 841 is 29.

The method that has been developed up to this point is easily implemented by typical digital techniques. For a machine similar to the EC 130, however, it will require too much time to be a practical method. The final step of dividing by 2, which can add as much as one second to the total time required, is to be avoided if possible. The following procedure can be used to eliminate this step.

The previous examples used successive odd integers in the subtraction process, as 1, 3, 5, 7, 9, etc. The final step in the process is the division by 2. The same end result can be produced by separating each of the above subtraction cycles into 2 steps. The table below shows how this can be done. Note that for each normal subtract cycle there are 2 subtract cycles in the modified system. The first half of the subtract pair uses the previous integer developed in the prior step. (In the first subtract cycle this is of course 0.) The second half of the subtract pair uses the same integer with 1 added, (incremented by 1).

Thus, in the modified method, the first subtraction is by zero, and the second subtraction is by 1, (0 + 1). The first half of the second pair subtracts a 2, (1 + 1), and so on.

NORMAL METHOD		MODIFIED METHOD			
Subtract cycles	Odd integer	Subtract cycles		Odd integer	
1	1	1st pair	1 2	(1)	0 0 + 1 = 1
2	3	2nd pair	3 4	(3)	1 1 + 1 = 2; 2 + 1 = 3
3	5	3rd pair	5 6	(5)	2 2 + 1 = 3; 3 + 2 = 5
4	7	4th pair	7 8	(7)	3 3 + 1 = 4; 4 + 3 = 7
5	9	5th pair	9 10	(9)	4 4 + 1 = 5; 5 + 4 = 9

Since there are twice as many subtract cycles as before, it is not necessary to divide by 2 at the final step. The last successful subtract IS the root by this method. A complete example to 4 significant digits in the root is given below to illustrate the way the method works.

```
                         181 in couplets
                01                                         81
Subtract }      00                      Subtract }         10 —(1 moved left one place)
 pair (1)}      01                       pair (1)}         71
Last     }      01 —— (add 1)                              11 — (add 1)
successful}     00 —— Remainder                            60
subtract }      01                      Subtract }         11
             (Overdraft)                 pair (3)}         49
              (Restore)                                    12 — (add 1)
                                                           37
                                        Subtract }         12
                                         pair (5)}         25
                                        Last     }         13 — (add 1)
                                        successful}        12 — Remainder
                                        subtract }         13
                                                       (Overdraft)
                                                        (Restore)
```

The last successful subtraction yielded the number 13, and the square root of 181 is 13 plus a decimal amount. This decimal amount can be found by continuing the subtractions using another couplet (00) and whatever remainder is left, in this case 12.

To illustrate, using remainder 12:

```
              Remainder      Couplet added
                        1200    (13 moved left one place)
Subtract }               130
 pair (1)}              1070
                         131 —— (add 1)
                         939
Subtract }               131
 pair (3)}               808
                         132 —— (add 1)
                         676
Subtract }               132
 pair (5)}               544
                         133 —— (add 1)
                         411
Subtract }               133
 pair (7)}               278
Last successful }        134 —— (add 1)
 subtract (pair)}        144 —— Remainder
                         134
                          10
                         135
                     (Overdraft)
                      (Restore)
```

The last successful subtraction yielded the number 134, which properly decimal aligned would be 13.4. A closer approximation of the square root of 181, then, is 13.4. If more decimal places are desired, then another couplet (00) is used with the remainder each time for each decimal place.

To illustrate, using remainder 144:

```
Remainder ------------┬┬┐ ┌┐------------ Couplet added
                      14400
Subtract  }           -1340 ------ (134 moved left one place)
pair (1)  }           13060
                      -1341 ------ (add 1)
                      11719
Subtract  }           -1341
pair (3)  }           10378
                      -1342 ------ (add 1)
                       9036
Subtract  }           -1342
pair (5)  }            7694
                      -1343 ------ (add 1)
                       6351
Subtract  }           -1343
pair (7)  }            5008
                      -1344 ------ (add 1)
                       3664
Subtract  }           -1344
pair (9)  }            2320
Last successful }     -1345 ------ (add 1)
subtract        }       975 ------ Remainder
                       1345
                   (Overdraft)
                    (Restore)
```

The last successful subtraction yielded the number 1345, which properly decimal aligned would be 13.45. A closer approximation of the square root of 181 then is 13.45. If more decimal places are desired, then another couplet (00) is used with the remainder to give another decimal place. Addition of couplets and using the remainder each time can be used to give any desired degree of decimal accuracy.

In the foregoing illustration of square root, the odd integers were found by successively using the <u>last</u> number successfully subtracted as the first step in each subtract pair cycle, then adding 1 for the second step of the cycle. The odd integer can also be found by adding 1 to the last number successfully subtracted in the <u>first</u> step of the new subtract cycle, and for the second step using the last number without adding 1. In other words, mathematically it does not matter in which step of the subtract pair that the 1 is added; the sum of the two steps will still result in the next odd integer.

To illustrate using 25:

```
 25      24            24      22
- 1     - 0           - 2     - 1
 24      24            22      21
  Subtract
    pair
    (1)                   (3)

 21      18            16      12
- 3     - 2           - 4     - 3
 18      16            12       9

    (5)                   (7)

  9       4
- 5     - 4
  4       0

    (9)
```

This is the method used in the EC-132 logic. In the first step of each subtract cycle 1 is effectively added to the last number successfully subtracted, without, however, affecting the retention of the last number. As the second step in each pair, the retained last number is subtracted. If both pairs of subtracts are successful, then 1 is directly added to the last successfully subtracted number, which then is used in the next subtract cycle. It is important to note that each subtract in a pair of subtracts must be successful before the number representing the last successful subtract is actually changed.

In general terms, the EC-132 handles the square root problem as follows.

The number begins in R1 and the Square Root key is pressed. First, R1, R2, R3, and R4 are shifted up to put the number in R2, then R1 and R0 are cleared out. R2 and R1 are then shifted to left twice to put a couplet into R1. Next, the subtract operation begins with the answer (the odd order integer) being developed in R0. The subtract cycle continues until there is an overdraft, then the overdraft is cleared out by addition, and another couplet is shifted into R1. For every couplet that is shifted into R1, R0 is shifted left once. Thus, there is always one digit in the answer for each couplet in the original number. The answer is thus developed, digit by digit, in R0. At the end of the problem, R3 and R4 are shifted down, and R0 is shifted up into R1; these operations are done simultaneously, so it takes only one pass through the delay line.